SOLUTIONS MANUAL

High School | Geometry

Exercise Solutions for Chapter Lessons & Reviews

Answers to Exercises For

GEOMETRY

Seeing, Doing, Understanding

MASTERBOOKS® CURRICULUM

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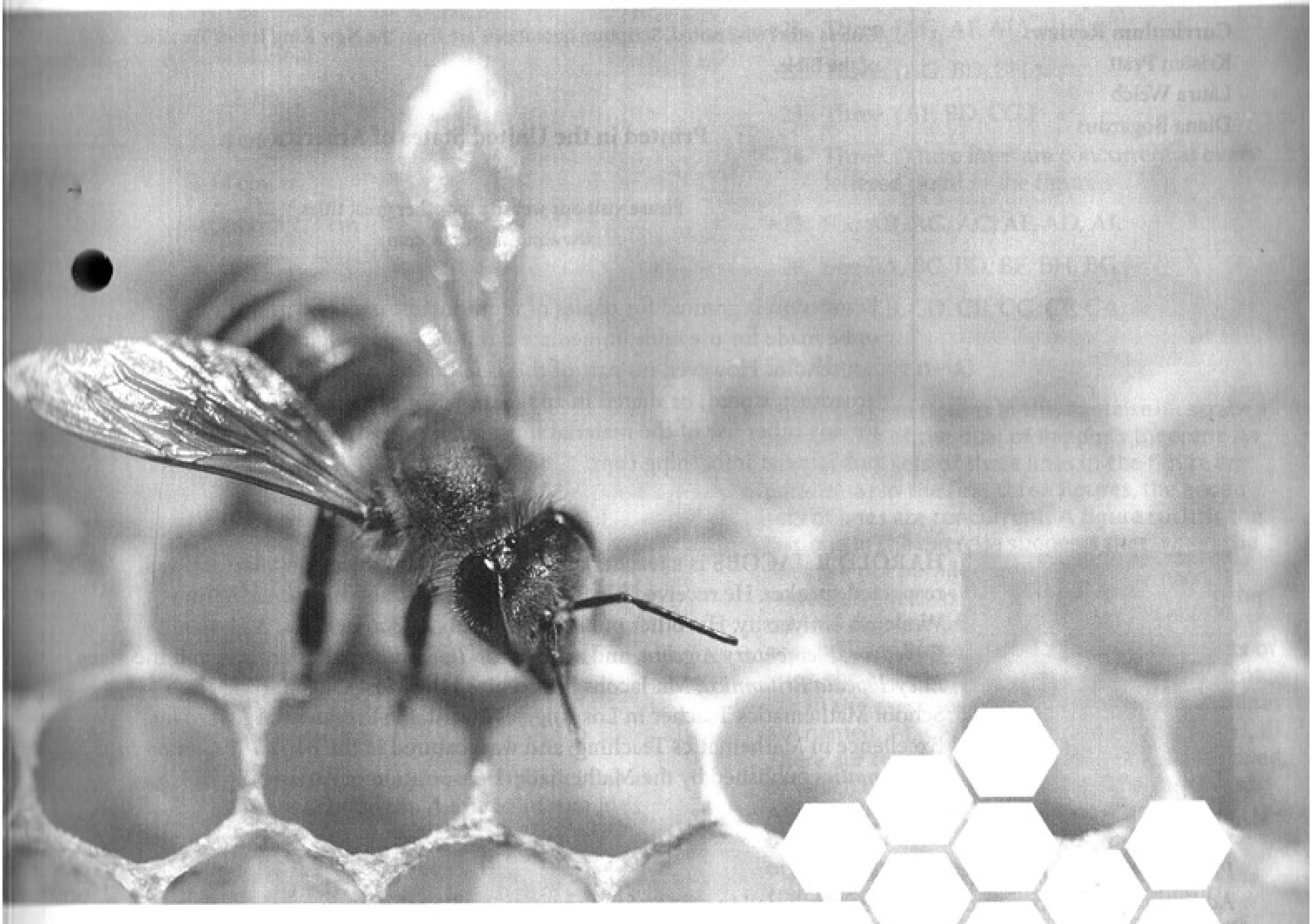

Harold R. Jacobs

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Author: Harold R. Jacobs

Master Books Creative Team:

Editor: Laura Welch

Cover Design: Diana Bogardus

Copy Editors:
Craig Froman
Willow Meek

Curriculum Review:
Kristen Pratt
Laura Welch
Diana Bogardus

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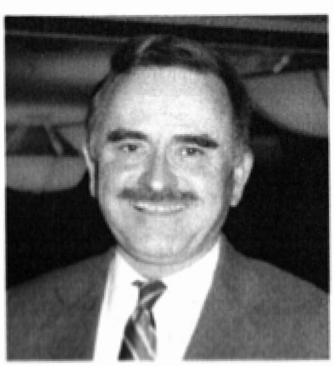


HAROLD R. JACOBS is a teacher of mathematics and science, writer, and well-respected speaker. He received his B.A. from U.C.L.A. and his M.A.L.S from Wesleyan University. His other publications include *Mathematics: A Human Endeavor*, *Elementary Algebra*, and articles for *The Mathematics Teacher* and the *Encyclopedia Britannica*. Mr. Jacobs has received the Most Outstanding High School Mathematics Teacher in Los Angeles award, the Presidential Award for Excellence in Mathematics Teaching, and was featured in the book *101 Careers In Mathematics* published by the Mathematical Association of America.

Publisher's Note:
Additional reference sources have been included in some of the answers in this solution manual to offer learning extensions for students if they wish to explore more on specific topics. The books represent a variety of worldviews, scientific disciplines, and perspectives. Use of these additional resources is not required for the course. Educators should use their discretion on incorporating these additional resources per the needs and requirements of their students and education program.

ANSWERS TO EXERCISES

Chapter 1, Lesson 1

Set I (pages 10–11)

Ch'ang-an, now called Sian, was one of two great capital cities of ancient China. It was built on the bank of the Wei River on a main trade route to Central Asia.

The city measured 5 miles from north to south and 6 miles from east to west and was surrounded by a wall 17.5 feet high. Its population in the eighth century is estimated to have been 2 million.

More about Ch'ang-an can be found in chapter 5 of *Ancient China,* by Edward H. Schafer (*Great Ages of Man* series, Time-Life Books, 1967).

1. Collinear.

•2. 12 cm.

3. 6 cm.

4. 4 cm.

•5. About 1.3 cm.

6. 10 cm.

•7. 3 mi. ($\frac{6 \text{ mi}}{2} = 3$ mi.)

8. 0.5 mi. ($\frac{6 \text{ mi}}{12 \text{ cm}} = \frac{0.5 \text{ mi}}{\text{cm}}$.)

9. 5 mi. (10 cm × $\frac{0.5 \text{ mi}}{\text{cm}}$ = 5 mi.)

10. No, because its sides are not all equal.

11. The colored regions (the Palace and Imperial Cities, the two markets, and the park).

12. 22 mi. (44 cm × $\frac{0.5 \text{ mi}}{\text{cm}}$ = 22 mi.)

13. Coplanar.

•14. Line segments.

15. A line segment is part of a line and has a measurable length. A line is infinite in extent.

16. 25.

• Answers to exercises marked with a bullet are also given in the textbook.

Set II (page 11)

The figure in this exercise set is a "nine-three" configuration, one of three such configurations in which nine lines and nine points are arranged so that there are three lines through every point and there are three points on every line. It and other configurations of projective geometry are discussed in chapter 3 of *Geometry and the Imagination,* by David Hilbert and Stefan Cohn-Vossen (A.M.S. Chelsea Publishing, 1990).

17. Nine.

18. Points that lie on a line.

19. Nine. They are E-F-I, A-B-G, A-D-I, G-H-I, B-C-D, B-E-H, D-F-H, A-C-F, C-E-G.

20. Lines that contain the same point.

•21. Three. (AG, AF, AI.)

22. Three. (AG, BD, BH.)

23. Three. (AF, BD, CG.)

24. Three. (Three lines are concurrent at every lettered point of the figure.)

•25. Six; AB, AG, AC, AF, AD, AI.

26. Six; BA, BC, BD, BE, BH, BG.

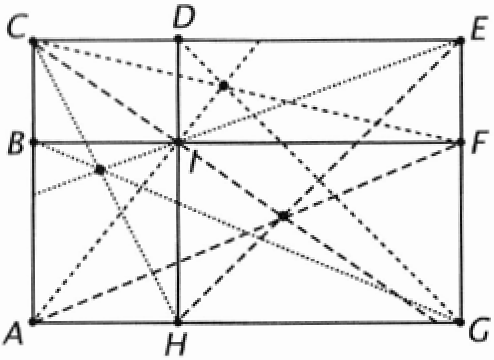
27. Six; CB, CD, CE, CG, CF, CA.

Set III (pages 11–12)

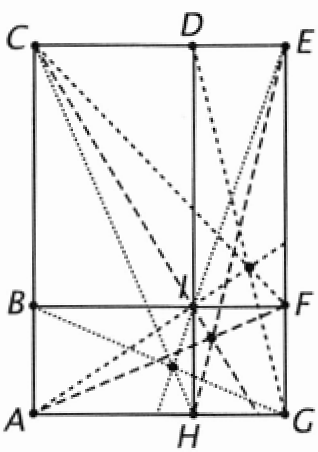
The first three figures in this exercise illustrate a special case of the dual of Pappus's theorem. As long as two sets of three lines in the figure are parallel, as in the first three figures, the added sets of lines are concurrent. A figure such as the last one in the exercise suggests that, when the lines are not parallel, the concurrence is lost. However, the two sets of three segments in the original figure must not necessarily belong to parallel lines. If they belong to concurrent sets of lines, the added sets of lines also are concurrent.

In the exercises, three sets of concurrent lines are named: AI, CF, DG; AF, CI, EH; and BG, CH, EI. There are three more sets that are not mentioned, of which one is AD, BE, GI. The other two sets, AE, BD, FH and BH, CG, DF, are more elusive (1) because, for certain spacings of the lines in the original figure, they are parallel rather than concurrent and (2) because, for the cases in which they *are* concurrent, the points of concurrency lie outside the figure.

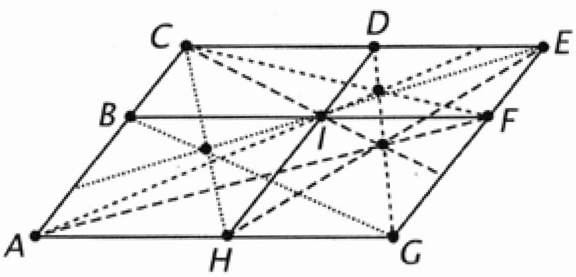
1.

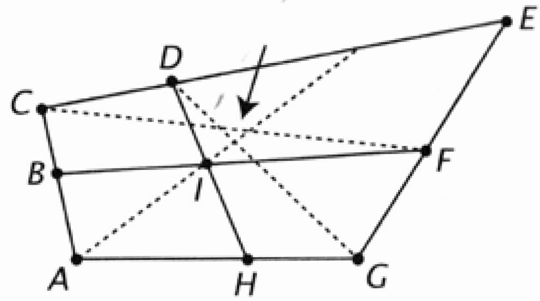


2. They seem to be concurrent.
3. These lines also seem to be concurrent.
4. These lines also seem to be concurrent.
5. *Example figure:*



6. The three sets of lines again seem to be concurrent.
7. *Example figures:*





(Just one set of the three lines is shown for the figure above, and the lines are clearly not concurrent.)

8. As long as the line segments in the original figure consist of two parallel (or concurrent) sets, the three sets of lines seem to be concurrent. Otherwise, they do not.

Chapter 1, Lesson 2

Set I (pages 15–16)

In the problems about the constellations, it is interesting to realize that, although the stars that form them give the illusion of being both coplanar and close together, they are certainly not coplanar and not necessarily near one another. The angles that are measured in the exercises are merely the angles between the lines in our map of the sky and not the actual angles between the lines in space.

Triangle Measurements.

1. AB.
•2. $\angle C$.
3. AC.
4. $\angle B$.
5. AB = 5.1 cm, AC = 2.6 cm, BC = 3.7 cm.
•6. Rays AC and AB.
7. $\angle A = 45°$.
8. *(Student drawing.)*
9. Rays BA and BC.
•10. $\angle B = 30°$.
11. Rays CA and CB.
12. $\angle C = 105°$.

Constellation Angles.

•13. Collinear.

14. *(Student guess.)* (∠B is actually smallest.)

15. *(Student guess.)* (∠2 is largest.)

•16. ∠1 = 67°.

17. ∠B = 60°.

18. ∠2 = 84°.

19. ∠R = 71°.

20. *(Student drawing.)*

21. ∠D = 143°.

•22. ∠I = 117°.

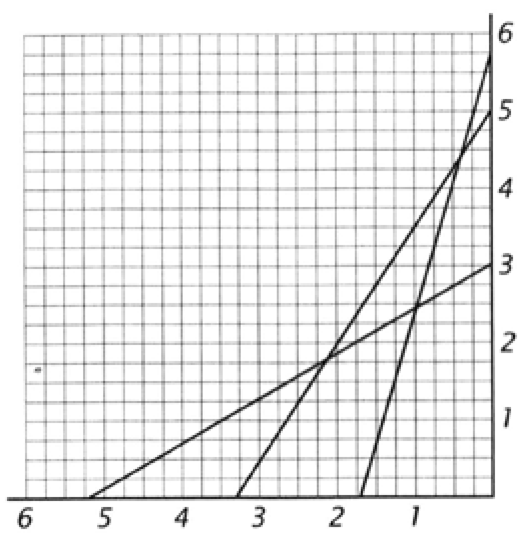
23. ∠P = 100°.

Set II (pages 16–17)

According to *Rules of Thumb,* by Tom Parker (Houghton Mifflin, 1983), "the base of a ladder should be 30 percent of its height from the base of the wall it is leaning against."

The Sliding Ladder.

24.



•25. Approximately $1\frac{3}{4}$ ft.

26. 74°.

27. 16°.

•28. Approximately $3\frac{1}{4}$ ft.

29. 56°.

•30. 34°.

31. Approximately $5\frac{1}{4}$ ft.

32. 30°.

33. 60°.

34. 45°. (The sum of the two angles seems to always be 90°.)

Set III (page 17)

More information on speculations about Mars can be found in the entertaining chapter titled "The Martian Dream" in *Pictorial Astronomy,* by Dinsmore Alter, Clarence H. Cleminshaw, and John G. Philips (Crowell, 1974).

Measuring Mars.

Dividing 24° into 360°, we get 15; so the distance around Mars is 15 times the distance between the two locations:

$$15 \times 880 \text{ miles} = 13{,}200 \text{ miles}.$$

Chapter 1, Lesson 3

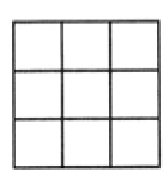
Set I (page 21)

Pentominoes were invented by Solomon Golomb, a long-time professor of mathematics and electrical engineering at the University of Southern California, who introduced them in a talk in 1953 at the Harvard Mathematics Club. His book titled *Polyominoes* (Princeton University Press, 1994) is the definitive book on the part of combinatorial geometry included in recreational mathematics. It is interesting to note that Arthur C. Clarke, the author of *2001—A Space Odyssey,* originally intended to use Pentominoes as the game played with the computer HAL before chess was substituted for it instead.

The problems about the Texas ranches were inspired by the fact that people sometimes naively assume that the size of a property can be measured by the time that it takes to walk around it. Proclus wrote about this assumption in 450 A.D. in his commentary on the first book of Euclid.

The Perimeter and Area of a Square.

1.



•2. 9 in^2.

•3. 12 in.

•4. 36 in^2.

•5. 24 in.

6. 1 ft^2.

7. 4 ft.

8. 144 in^2.

9. 48 in.

10. x^2 square units.

11. $4x$ units.

Pentominoes.

12. A polygon is a two-dimensional figure that is bounded by line segments.

•13. Piece I because it has four sides.

14. 5 square units.

15. Piece P. (It has a perimeter of 10 units; all the other pieces have perimeters of 12 units.)

Texas Ranches.

•16. Ranch A. (Ranch A needs 36 miles of fencing, whereas ranch B needs 32 miles).

17. Ranch B. (Ranch B has an area of 60 square miles, whereas ranch A has an area of 56 square miles).

18. Area.

Set II (page 22)

A book filled with fascinating information about the Great Pyramid is *Secrets of the Great Pyramid,* by Peter Tompkins (Harper & Row, 1971).

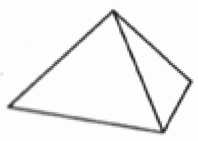
A Model of the Great Pyramid.

19. 64°.

20. Yes. (The sides opposite the 58° angles appear to be about 9.4 cm each.)

21. *(Model of Great Pyramid.)*

22. *Example figure:*



•23. Polygons.

24. A polyhedron.

•25. Eight.

26. Five.

27. Four.

•28. Three.

29. Five.

•30. A line.

31. *(Student estimate.)* (Calculations using trigonometry give a height of about 6.25 cm.)

•32. 75.6 ft.

33. *(Student estimate.)* (6.25 × 75.6 gives a height of approximately 470 ft.)

34. A view from directly overhead.

Set III (page 23)

The square-to-triangle puzzle was created, as the footnote on page 23 states, by Henry Dudeney, England's greatest inventor of mathematical puzzles. In *The Canterbury Puzzles,* he told of exhibiting it to the Royal Society of London in 1905 in the form of a model made of four mahogany pieces hinged together in a chain. When closed in one direction, the pieces formed the triangle and, when closed in the other direction, they formed the square. The version illustrated on page 23 of the text is slightly altered to make it easier for beginning geometry students to draw. Martin Gardner gives directions for constructing the figure exactly on pages 33and 34 of *The 2nd Scientific American Book of Mathematical Puzzles and Diversions* (Simon & Schuster, 1961).

A Square Puzzle.

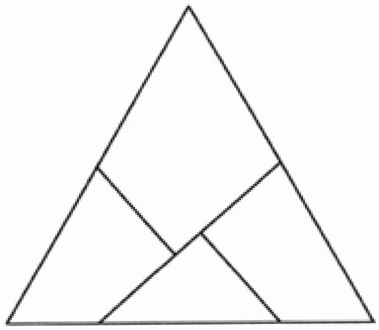
1. 60°.

2. ∠BFP = 79° and ∠AEH = 49°.

3. ∠BEH = 131° and ∠EHA = 41°.

4. PF and PG.

5.



6. About 21.3 cm.

7. 60°.

8. The area of the triangle is equal to that of the square because they are made of the same pieces.

9. 196 cm^2.

10. It is longer. (The perimeter of the triangle is about 64 cm, whereas the perimeter of the square is 56 cm.)

Chapter 1, Lesson 4

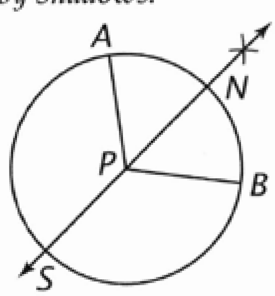
An excellent history and mathematical treatment of sundials is *Sundials—Their Theory and Construction*, by Albert E. Waugh (Dover, 1973).

The most common type of sundial is the horizontal one and is the kind depicted on the Kellogg's Raisin Bran box. For this type of sundial to tell time accurately, the angle of its gnomon must be equal to the latitude of the location in which the dial is used. The angle of the gnomon printed on the Kellogg box has a measure of about 39°, evidently chosen because that is a good approximation of the average latitude of the United States. Seattle has a latitude of almost 48°, whereas Miami's latitude is less than 26°; in locations such as these, the claim that "now you can tell time outdoors" with the cereal box sundial is less than accurate!

Set I (pages 26–27)

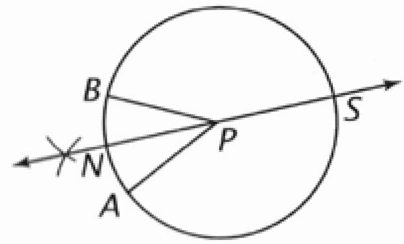
Finding North by Shadows.

1.



2. *(Student answer.)* (∠APB should be twice ∠APN.)

3.

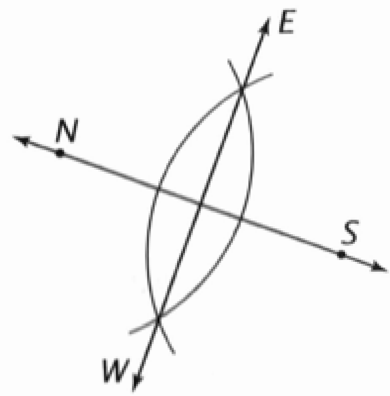


4. *(Student answer.)* (∠APB should be twice ∠APN.)

•5. Noncollinear.

6. Coplanar.

7.

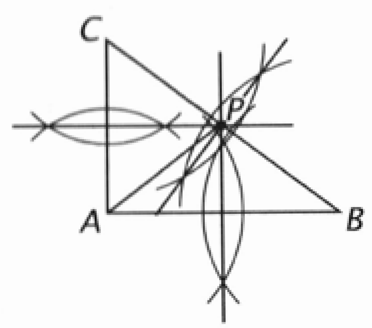


8. *(Student answer.)* (The angles formed by NS and EW have measures of 90°.)

9. *(Student answer.)* (The sum of all four angles is 360°.)

Bisectors in a Triangle.

10.

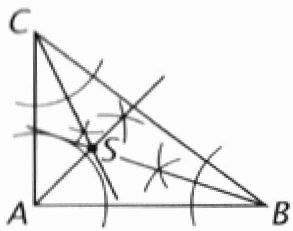


•11. The bisecting lines seem to be concurrent.

12. They seem to meet on side CB.

13. The distances seem to be equal.

14.



15. They seem to be concurrent.

•16. No.

Set II (pages 27–29)

About Sundials.

•17. West.

18. North.

19. South.

20. Noon.

•21. About 6 A.M. and 6 P.M.

22. ∠IOJ.

•23. OI bisects ∠HOJ.

24. No. ∠POQ and ∠QOR do not appear to be equal.

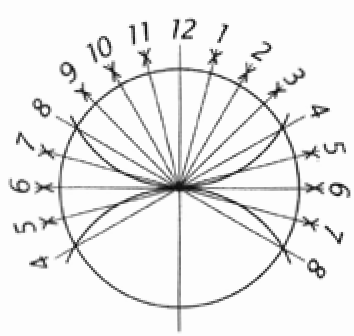
•25. ∠DOB and ∠EOA.

26. ∠POL.

27. ∠COB, ∠PON and ∠POQ.

28. ∠EOM.

29. *Example figure:*



30. Collinear means that there is a line that contains the points.

•31. 60°.

32. 30°.

•33. 180°.

34. 90°.

35. 15°.

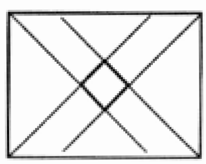
36. 4 minutes. ($\frac{60 \text{ minutes}}{15°} = \frac{4 \text{ minutes}}{1°}$.)

Set III (page 29)

This discovery exercise is adapted from a proof exercise (3.5.18) in *Geometry Civilized,* by J. L. Heilbron (Oxford University Press, 1998). Every geometry teacher should have a copy of this beautifully written and illustrated book. Dr. Heilbron was formerly Professor of History and the History of Science and Vice Chancellor of the University of California at Berkeley and is currently Senior Research Fellow at Worcester College, Oxford University, and at the Oxford Museum for the History of Science.

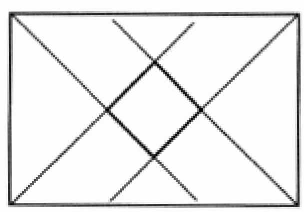
Angle Bisectors in a Rectangle.

1.



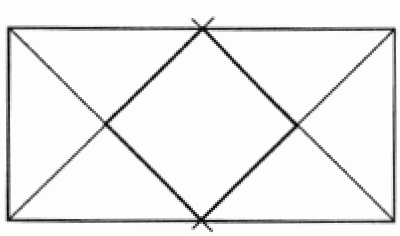
2. A square.

3.



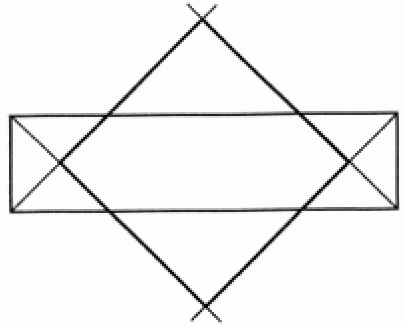
Again, the lines seem to form a square.

4.



The lines seem to form a square.

5.



The lines still seem to form a square.

6. Yes. The more elongated the rectangle, the larger the square.

7. If the original rectangle were square in shape, the polygon would shrink to a point.

Chapter 1, Lesson 5

Set I (pages 31–32)

The incorrect Egyptian formula for finding the area of a quadrilateral was evidently based on the notion of multiplying the averages of the lengths of the opposite sides. It is easy to show (as the students will be asked to do later in the course) that this method gives the correct answer if the quadrilateral is a rectangle. For all other quadrilaterals, the method gives an area that is too large.

The Egyptian Tax Assessor.

•1. 100 square units. ($\frac{1}{4}20 \cdot 20 = 100.$)

2. 88 square units. ($\frac{1}{4}16 \cdot 22 = 88.$)

3. 98 square units. ($\frac{1}{4}14 \cdot 28 = 98.$)

4. Ramses.

5. Yes. ($11 \cdot 8 = 88.$)

•6. It contains 80 square units. ($10 \cdot 8 = 80.$)

7. It contains 108 square units. ($9 \cdot 12 = 108.$)

•8. 54 square units. ($\frac{1}{2}108 = 54.$)

9. It contains 60 square units. ($5 \cdot 12 = 60.$)

10. 30 square units. ($\frac{1}{2}60 = 30.$)

•11. It contains 84 square units. ($54 + 30 = 84.$)

12. Ramses is cheated the most. He pays the highest tax, yet actually has the smallest property. (See the chart below.)

	Assessment Area	*Actual Area*
Ramses	100	80
Cheops	88	88
Ptahotep	98	84

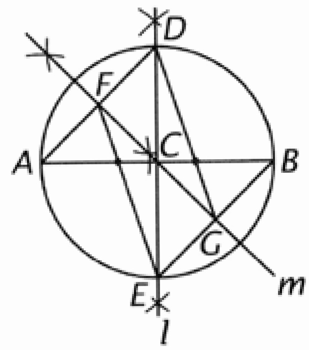
Set II (pages 32–34)

We will return to the trisection of line segments and angles later in the course. The method for trisecting an angle presented here is surely the first one to be thought of by most people encountering the problem for the first time. For small angles, it is a fairly good approximation but, for an angle of 60°, the error just begins to become detectable through measurement and, for obtuse angles, its failure becomes very obvious.

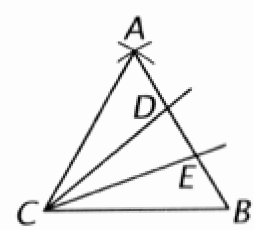
The classic book on angle trisection for many years was *The Trisection Problem,* by Robert C. Yates, originally published in 1942 and reprinted in 1971 by N.C.T.M. as a volume in the *Classics in Mathematics Education* series. The definitive book is currently *A Budget of Trisections,* by Underwood Dudley (Springer Verlag, 1987). Dr. Dudley includes some additional information on the subject in "Trisection of the Angle" in his book titled *Mathematical Cranks* (M.A.A., 1992). All make fascinating reading.

Trisection Problems.

13.



14.



15. An accurate drawing and careful measurements suggest that ∠ACD, ∠DCE, and ∠ECB are about 19°, 22°, and 19°, respectively.

Set III (page 34)

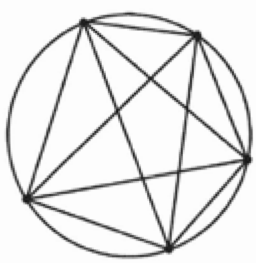
This problem was discovered by Leo Moser, who, in a letter to Martin Gardner, said that he thinks it was first published in about 1950. The "obvious" formula for the number of regions, $y = 2^{n-1}$, where n is the number of points, is wrong. The correct formula,

$$y = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$$

is surprisingly complicated. For more on this problem, see the chapter titled "Simplicity" in *Mathematical Circus* by Martin Gardner (Knopf, 1979).

Cutting Up a Circle.

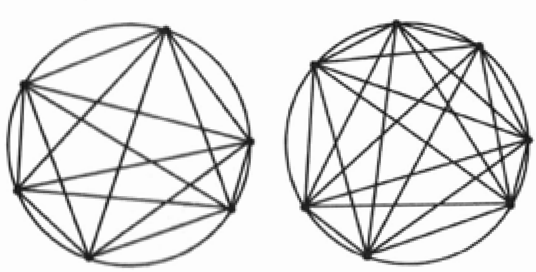
1. *Example figure:*



2. 16.
3. 2.
4. 4.
5.

No. of points connected	2	3	4	5
No. of regions formed	2	4	8	16

6. It doubles.
7. 32.
8. 64.
9. *Example figures:*



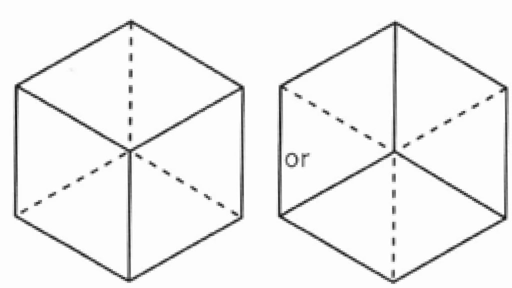
10. No. (Six points result in at most 31 regions, and seven points in at most 57 regions.)

Chapter 1, Review

Set I (pages 36–37)

The transparent view of the cube used in exercises 15 through 17 is named after psychologist Hertha Kopfermann, who studied it in 1930. An interesting discussion of our perception of this figure appears on pages 27 and 28 of *Visual Intelligence,* by Donald D. Hoffman (Norton, 1998).

1. Line segments.
•2. Polygons (or squares).
3. A polyhedron.
4. 3.
•5. 12.
6. 3.
7. 6.
•8. Coplanar.
9. 9.
•10. PQ.
11. NO.
12. OP.
•13. T.
14. U and Q.
15. Triangles, quadrilaterals, and a hexagon. (Students who are aware of concave polygons may notice more than one hexagon and even some heptagons!)
16.



17. 3.

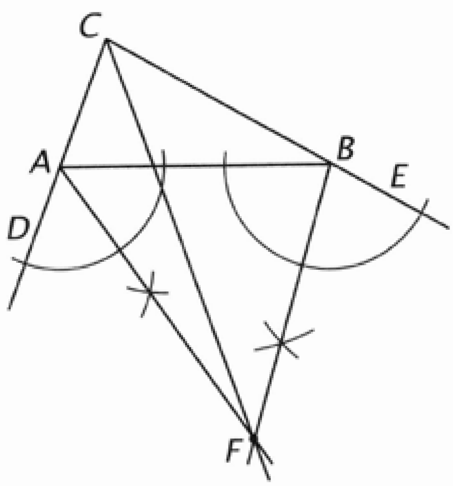
Set II (pages 37–38)

Exercises 19 through 26 are based on the fact that every triangle has three excenters as well as an incenter. Each excenter is the point of concurrence of the lines bisecting exterior angles at two vertices of the triangle (∠DAB and ∠ABE in the figure) and the line that bisects the (interior) angle at the third vertex (∠C in the figure). Point F is one of the excenters of the triangle.

Exercises 27 through 30 are based on the fact that, when the midpoints of the sides of any quadrilateral are connected in order with line segments, the quadrilateral that results is a parallelogram. Furthermore, its perimeter is equal to the sum of the length of the diagonals of the original quadrilateral. These facts are easily proved and will be established later in the course.

•18. Constructions.

19.



•20. ∠DAB = 110°.

21. ∠ABE = 150°.

22. ∠DCE = 80°.

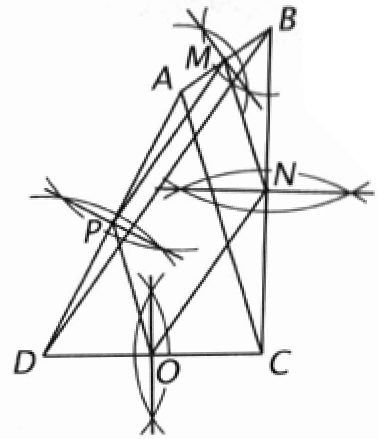
23. ∠DAF = ∠FAB = 55°.

24. ∠ABF = ∠FBE = 75°.

•25. ∠DCF = ∠FCE = 40°.

26. Line CF seems to bisect ∠DCE.

27.



28. The opposite sides of the quadrilateral appear to be equal (and parallel).

29. (Answer depends on student's drawing.)

30. (Answer depends on student's drawing but should be close to the answer to exercise 29.)

Set III (page 38)

The story of Queen Dido and Carthage is told on pages 44–47 of *Mathematics and Optimal Form*, by Stefan Hildebrandt and Anthony Tromba (Scientific American Library, 1985) and on pages 64–71 of the revised and enlarged version of this book titled *The Parsimonious Universe* (Copernicus, 1996).

•1. 1,200 yd. ($4 \cdot 300 = 1{,}200$.)

2. 90,000 yd². ($300^2 = 90{,}000$.)

3. 500 yd. ($900 - 2 \cdot 200 = 500$.)

4. 1,400 yd. ($2 \cdot 200 + 2 \cdot 500 = 1{,}400$.)

5. 100,000 yd². ($200 \cdot 500 = 100{,}000$.)

6. *(Student drawings.)* (Students who have made several drawings may suspect that the length along the river should be twice the side width to get as much land as possible. In this case, the numbers are 450 yards and 225 yards and the area is 101,250 square yards.)

Chapter 1, Algebra Review (page 40)

•1. Associative for addition.

•2. Commutative for multiplication.

•3. Definition of subtraction.

•4. Identity for addition.

•5. Definition of division.

•6. Inverse for multiplication.

•7. 1,728. ($12 \cdot 12 \cdot 12 = 1{,}728$.)

•8. 12,000. ($12 \cdot 10 \cdot 10 \cdot 10 = 12{,}000$.)

•9. 20. ($5 + 15 = 20$.)

•10. –75.

•11. 225. ($289 - 64 = 225$.)

•12. 81. ($9^2 = 81$.)

•13. 49π.

•14. 14π.

•15. $5x$.

•16. x^5.

•17. $5x$.

•18. x^5.

•19. $2x + 3y$.

•20. x^2y^3.

•21. $x^2 + y^3$.

•22. $x^3 + y^2$.

•23. $x + 2y$.

•24. xy^2.

•25. $7x^2$.

•26. $7x^2$.

•27. $5x^2$.

•28. $5x^2$.

•29. $4x^2 + 3x$.

•30. $3x^3 + 2x^2 + x$.

•31. x^{14}.

•32. x^{14}.

•33. $6x^5$.

•34. $6x^5$. ($2x^3 \cdot 3x^2 = 6x^5$.)

•35. $36x^2$. ($6x \cdot 6x = 36x^2$.)

•36. $72x^5$. ($8x^3 \cdot 9x^2 = 72x^5$.)

•37. $4x + 4y$. ($3x + 2y + x + 2y = 4x + 4y$.)

•38. $4x + 4y$.

•39. $2x$. ($3x + 2y - x - 2y = 2x$.)

•40. $2x$.

•41. 0.

•42. 1.

•43. $5x + 10y$.

•44. $3x + 4y$.

•45. $5x - 10y$.

•46. $3x - 4y$.

•47. $31xy$. ($28xy + 3xy = 31xy$.)

•48. $25xy$. ($28xy - 3xy = 25xy$.)

•49. $7 + x^3 + x^5$.

•50. $10x^8$.

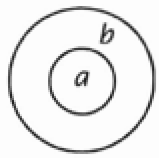
Chapter 2, Lesson 1

Set I (pages 43–44)

•1. A conditional statement.

2. "*a*" represents the hypothesis and "*b*" represents the conclusion.

3.



A circle labeled *a* inside another circle labeled *b*.

•4. You live in the Ozarks.

5. You live in the United States.

•6. Yes.

7. No.

8. No. If it is cold outside, it isn't necessarily snowing.

9. No.

10. It is cold outside if it is snowing.

11. Yes.

12. Statements 1 and 3.

13. Yes.

•14. If an animal is a koala bear, then it eats only eucalyptus leaves.

15. If the cat is in the birdcage, then it isn't there to sing.

16. If money grew on trees, then Smokey Bear wouldn't have to do commercials for a living.

17. If you are an architect, then you use geometry.

•18. If I don't understand, then I ask questions.

19. If there is a fire, then use the stairs instead of the elevator.

20. If you are an elephant, then you cannot fly. (Or, If you can fly, then you are not an elephant.)

Set II (pages 44–45)

•21. If at first you don't succeed, then try again.

22. Try again if at first you don't succeed.

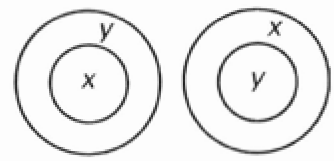
23. Region 1.

•24. Regions 1 and 2.

25. Regions 2 and 3.

26. Region 3.

27.



If *x*, then *y*   If *y*, then *x*

•28. The second.

29. The first.

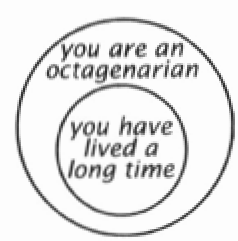
30. Two.

31.

32. Statements a and d.

33. Statements a and d.

34.



•35. Statements c and d.

36. Statements a and b.

Set III (page 45)

The Animal Mind, the source of the information for the Set III exercises, is a fascinating book. This paragraph from the dust jacket describes it well:

"In this engaging volume, James and Carol Gould go in search of the animal mind. Taking a fresh look at the evidence on animal capacities for perception, thought, and language, the Goulds show how scientists attempt to distinguish actions that go beyond the innate or automatically learned. They provide captivating, beautifully-illustrated descriptions of a number of clever and curious animal behaviors—some revealed to be more or less pre-programmed, some seemingly proof of a well-developed mental life."

1. The red monkey shape.
2. In the left-hand column.
3. The red square.
4. If Sarah takes the banana, then Mary will *not* give chocolate to Sarah.

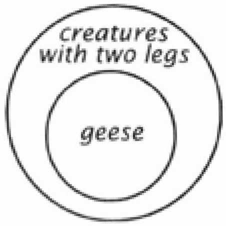
Chapter 2, Lesson 2

Set I (pages 47–48)

Exercises 1 through 4 have an interesting source. In his book titled *Euclid and His Modern Rivals* (1885), Charles L. Dodgson (Lewis Carroll) criticized some of the geometry books of the time. One of them, *Elementary Geometry, following the Syllabus prepared by the Geometrical Association,* by J. M. Wilson (1878) contained on page 17 the following "question":

"State the fact that all 'geese have two legs' in the form of a Theorem." Carroll wrote: "This I would not mind attempting; but, when I read the additional request, to 'write down its converse *theorem*,' it is so powerfully borne in upon me that the writer of the Question is probably himself a biped, that I feel I must, however reluctantly, decline the task."

1. If a creature is a goose, then it has two legs.
2.



•3. If a creature has two legs, then it is a goose.

4. No.

•5. You are not more than six feet tall.

6. If you are not more than six feet tall, then you are an astronaut.

7. No.

8. No.

•9. If you do not know how to reason deductively.

10. If you cannot comprehend geometry, then you do not know how to reason deductively.

11. You do not know how to reason deductively.

12. If and only if.

•13. They have the same meaning.

14. If and only if.

15. The fear of peanut butter sticking to the roof of your mouth.

16. If you are afraid of peanut butter sticking to the roof of your mouth, then you have arachibutyrophobia.

17. Yes. It must be true because the sentence is a definition.

18. Common, proper, improper.

19. If a fraction is a vulgar fraction, then it is a common fraction that is either proper or improper.

20. If a fraction is a common fraction that is either proper or improper, then it is a vulgar fraction.

•21. Yes. The converse must be true because this is a definition.

22. "New Year's Day" does not mean the same thing as "holiday." Other days are holidays as well, so its converse is not true.

23. Yes. "New Year's Day" means the same thing as "the first day of the year." The converse is true.

Set II (pages 48–49)

The wolf-pack definition in exercises 36 and 37 is an amusing example of recursion. An entire chapter of Clifford Pickover's book titled *Keys to Infinity* (Wiley, 1995) deals with the subject of "recursive worlds."

24. Yes.

25. No.

•26. No.

27. The second sentence. (Only if it is your birthday do you get some presents.)

•28. It is the converse.

29.

30. It is dry ice if and only if it is frozen carbon dioxide.

31. The teacher thought you misspelled "if."

•32. Sentence 2. (If it is a detective story, then it is a whodunit.)

33. Yes.

34. If a car is a convertible, then it has a removable top. If a car has a removable top, then it is a convertible.

•35. It is the converse.

36. Two or more.

37. Because, according to the definition, there is no limit to the number of wolves in a wolf pack.

Set III (page 49)

Since its creation in the early 1970s, Peter Wason's puzzle has appeared in puzzle books in a variety of forms. In *The Math Gene* (Basic Books, 2000), Keith Devlin remarks that most people find the puzzle extremely hard.

Turn over the blue card to see if there is a circle on the other side. Turn over the card with the triangle to see if the other side is blue. (It doesn't matter what is on the other side of the card with the circle. For example, the possibility that the other side is red doesn't contradict the statement

> If a card is blue on one side, it has a circle on the other side.

Chapter 2, Lesson 3

Set I (pages 52–53)

1. If you keep quiet, others will never hear you make a mistake.
   If others never hear you make a mistake, they will think you are wise.

•2. If you keep quiet, others will think you are wise.

3. $a \to b$, $b \to c$. Therefore, $a \to c$.

•4. All Greeks are humans.
   All humans are mortals.
   Therefore, all Greeks are mortals.

5. All Greeks are statues.
   All statues are mortals.
   Therefore, all Greeks are mortals.

6. Yes.

•7. No.

•8. The hypothesis.

9. The conclusion.

10. $a \to b$, $b \to c$, $c \to d$. Therefore, $a \to d$.

11. The third: If you live where it is cold, you see a lot of penguins.

•12. It also may be false.

13. If Captain Spaulding is in the jungle, he can't play cards.

14. Yes.

15. If NASA launched some cows into space, they would be the herd shot around the world.

•16. A theorem.

17. If you go to Dallas, you will take a plane.
If you take a plane, you will go to the airport.
If you go to the airport, you will see all the cabs lined up.
If you see all the cabs lined up, you will see the yellow rows of taxis.

•18. If you go to Dallas, you will see the yellow rows of taxis.

19. If a duck had sore lips, he would go to a drugstore.
If he went to a drugstore, a duck would ask for some Chapstick.
If he asked for some Chapstick, a duck would ask to have it put on his bill.

20. *Second statement:* If they wanted to take along some food, they would try to carry on six dead raccoons.
Last statement: If the flight attendant objected, they would be told that there is a limit of two carrion per passenger.

21. *First statement:* If a group of chess players checked into a hotel, they would stand in the lobby bragging about their tournament victories.
Third statement: If the manager asked them to leave, they would ask why.

Set II (pages 53–54)

Dice Proof.

•22. 21. (1 + 2 + 3 + 4 + 5 + 6 = 21.)

23. 7. ($\frac{21}{3} = 7$.)

24. 2.

25. 6.

Matchbox Proof.

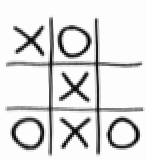
•26. Red. Because the label is wrong.

27. A red marble and a white marble.

28. Two white marbles.

Tick-tack-toe Proofs.

29.



30. If your opponent marks an X in the upper left, then you will mark an O in the lower right.
If you mark an O in the lower right, then your opponent will mark an X in the lower center.
If your opponent marks an X in the lower center, you will mark an O in the upper center.

31. *(Student drawings.)*

32. You will win the game because, no matter where you mark your next O, your opponent will then be forced to allow you to set a winning trap!

Set III (page 54)

Word ladders were invented by Lewis Carroll in 1877 and have been a popular form of word play ever since. An entire chapter in Martin Gardner's *The Universe in a Handkerchief—Lewis Carroll's Mathematical Recreations, Games, Puzzles, and Word Plays* (Copernicus, 1996) deals with word ladders (called "doublets" by Carroll).

Another interesting reference is *Making the Alphabet Dance,* by Ross Eckler (St. Martin's Press, 1996). At the beginning of chapter 4, titled "Transforming One Word into Another", Eckler remarks: "By drawing on ideas from mathematics . . . [word play] gains additional respectability and legitimacy."

1. *Example solution:*
LESS, LOSS, LOSE, LOVE, MOVE, MORE.

2. *Example solution:*
HEAD, HEAR, HEIR, HAIR, HAIL, TAIL.

Chapter 2, Lesson 4

Set I (pages 57–58)

It is interesting to note that, like that of Rube Goldberg, Samuel Goldwyn's name has entered the dictionary. According to *The Random House Dictionary of the English Language,* a "Goldwynism" means "a phrase or statement involving a humorous and supposedly unintentional misuse of idiom."

•1. Contradiction.

2. "In–direct."

•3. Suppose that it would not speak foul language.

4. Suppose that he has control over his pupils.
5. Suppose that it does not lead to a contradiction.

•6. No.

7. Yes.

8. No.

9. Indirect.

•10. It is the opposite of the theorem's conclusion.

•11. The earth is not flat.

12. Suppose that the earth is flat.

13. The stars would rise at the same time for everyone, which they do not.

•14. That it is false.

Set II (pages 58–59)

The puzzles about the weights are adapted from ideas in A. K. Dewdney's book titled *200% of Nothing* (Wiley, 1993). The subtitle of the book is *An Eye-Opening Tour through the Twists and Turns of Math Abuse and Innumeracy.* Dewdney uses the idea of equal partition of numbers in a section of the book titled "What Do Mathematicians Do?"

Trading Desks Proof.

•15. *Beginning assumption:*
Suppose that the pupils can obey the teacher.
Contradiction:
There are 13 pupils at the black desks but only 12 brown desks.
Conclusion:
The pupils can't obey the teacher.

Ammonia Molecule Proof.

16. *Beginning assumption:*
Suppose that the atoms of an ammonia molecule are coplanar.
Contradiction:
Each bond angle is 107°.
Conclusion:
The atoms of an ammonia molecule are not coplanar.

Rummy Proof.

17. *Beginning assumption:*
Suppose that someone holds a royal flush.
Contradiction:
You have all the "10" cards.
Conclusion:
In this deal of the cards, no one holds a royal flush.

Balanced Weights Proofs.

18. Puzzle 1. One set is 2, 4, 6, 13; the other set is 7, 18.

19. If a puzzle of this type has a solution, then each set will weigh half the total weight.

20. *Beginning assumption:*
Suppose that there is a solution.
Contradiction:
The sum of all of the weights is odd.
Conclusion:
There is no solution.

Athletes Puzzle.

21. There is only 1 football player and consequently there are 99 basketball players. If there were two or more football players, then, given any two of the athletes, both could be football players. This contradicts the fact that, given any two of the athletes, at least one is a basketball player. Therefore, the assumption that there are two or more football players is false, and so there is only one football player.

Set III (page 59)

A good discussion of David L. Silverman's puzzle about the list of true and false statements appears on page 79 of *Knotted Doughnuts and Other Mathematical Entertainments,* by Martin Gardner (W. H. Freeman and Company, 1986). In its original form, the puzzle contained 1,969 statements (to celebrate the new year of 1969) comparable to those in the list in Set III. The only true statement in the original version was the 1,968th one.

This sort of puzzle suggests other possibilities that students might enjoy exploring. What if the word "false" were changed into "true" in all of the statements? What if the word "exactly" were omitted?

1. No, because any two statements on the list contradict each other.

2. No, because then the list would contain exactly zero false statements.

3. No.

4. No.

5. One, because the preceding conclusions eliminate all of the other possibilities. The true statement must be C.

Chapter 2, Lesson 5

Set I (pages 61–63)

The dog Rin-Tin-Tin was Warner Brothers' first big money-making star. His movie career began in 1922, and his salary at one time was $1,000 a week. According to Clive Hirschhorn's *The Warner Brothers Story,* Rin-Tin-Tin's 19 films kept the studio prosperous during the silent era and saved many theaters from closure. He was voted most popular film performer in 1926!

1. No (because a period has dimensions.)

2. No (because even an atom has dimensions.)

•3. You go around in circles.

4. Star, tail.

5. Because Rin-Tin-Tin was a movie star who had a tail.

•6. Two points determine a line.

7. Because it is taut and hence straight like a line.

•8. Theorems.

9. A statement that is assumed to be true without proof.

10. No.

Three Arrow Table.

11. Three noncollinear points determine a plane.

•12. The points.

13. The plane.

14. The table top might tip over.

15. Noncollinear.

16. If and only if.

17. If points are coplanar, then there is a plane that contains all of them. If there is a plane that contains all of them, then points are coplanar.

•18. Converse.

Intersecting Cubes.

19. True.

•20. True.

21. True.

•22. False.

23. False.

24. True.

25. True.

Set II (pages 63–64)

Japanese Weights.

•26. A picul is 1,600 taels.

27. A picul is 16,000 momme.

28. A momme is one-tenth of a tael.

29. No, because we don't know what any of these units are.

30. Yes. An elephant weighs more than one picul.

31. Because "the load carried by a man" is too vague; some men can carry much heavier loads than others.

Credit Card Rules.

•32. Statements 1 and 3.

33. Transaction finance charge and supercheck.

34. Statements 2, 4, and 5.

35. Statements 2 and 4. (If you go over your credit limit, you will be charged a fee. If you are charged a fee, the fee will be added to your balance.)

36. If you go over your credit limit, a fee will be added to your balance.

Crooked Lines.

•37. Two points determine a line.

38. Yes.

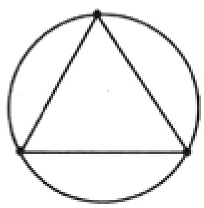
39. We did not define the word "line" (nor did we define the word "straight").

40. A theorem is a statement that is proved.

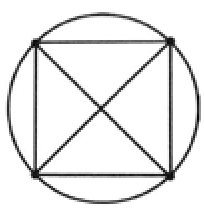
41. *Beginning assumption:*
Suppose two lines can intersect in more than one point.
Contradiction:
Two points determine a line.
Conclusion:
Two lines cannot intersect in more than one point.

Set III (page 64)

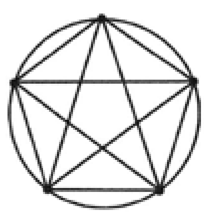
1. 3 lines.



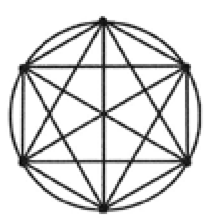
2. 6 lines.



3. 10 lines.



4. 15 lines.



5. 20 points: 190 lines.

Some students who answer this correctly will probably do it by observing the pattern of increasing differences.

Number of points	2	3	4	5	6
Number of lines	1	3	6	10	15
	2	3	4	5	

Each additional point results in one more line than the point before it.

Other students may deduce a formula for the number of lines, *l*, in terms of the number of points, *p*:

$$l = \frac{p(p-1)}{2}$$

(This is the formula for the binomial coefficient: $\binom{p}{2}$.)

Chapter 2, Lesson 6

Set I (pages 67–68)

Sherman Stein makes some scathing remarks in his book titled *Strength in Numbers* (Wiley, 1996) about the "tear off the corners" of a triangle procedure illustrated in exercises 8 through 10.

He writes:

"Here is how one of the seventh-grade texts has the pupils 'discover' that the sum of the three angles of a triangle is always 180°. Under the heading 'Work Together,' it has these directions:

Begin with a paper triangle that is a different shape from those of the other members of your group. Number the angles of the triangle and tear them off the triangle. Place the three angles side-by-side so that pairs of angles are adjacent and no angles overlap.

What seems to be true? Compare your results with those of your group. Make a conjecture.

On the very next page, facing these directions, we find, 'In the Work Together activity you discovered that the following statement is true.' In boldface follows, 'The sum of the measures of the angles of a triangle is 180°.'

What kind of discovery or constructing knowledge out of experience is that? From what I know about pupils, they would glance at the bold type on the next page and stop experimenting."

•1. The length of the hypotenuse of a right triangle.

2. The speed of light.

3. The circumference of a circle.

4. The length of one of the other sides of a right triangle.

•5. The area of a circle.

6. The Pythagorean Theorem.

7. The squares on the sides contain 16, 9, and 25 small squares; $16 + 9 = 25$.

8. The sum of the angles of a triangle is 180°.

•9. No.

10. No.

Target Circles.

11. 2, 4, 6, 8, and 10 units.

•12. 2π, 4π, 6π, 8π, and 10π units.

•13. 9π square units. ($\pi 3^2 = 9\pi$.)

14. 9π square units. ($\pi 5^2 - \pi 4^2 = 9\pi$.)

15. It is because the purple region appears to many people to be much larger than the yellow region.

16. 100. $36 + 64 = 100$ by the Pythagorean Theorem.

17. 144. $169 - 25 = 144$ by the Pythagorean Theorem.

•18. 30°. (The Triangle Angle Sum Theorem.)

19. 45°.

20. $(90 - n)°$.

•21. 70°. ($180 - 40 = 140$, $\frac{140}{2} = 70$.)

22. 65°. ($180 - 50 = 130$, $\frac{130}{2} = 65$.)

23. $\left(\frac{180 - n}{2}\right)°$.

•24. No.

25. This is the converse of the Pythagorean Theorem, and the converse of a true statement isn't necessarily true. (We will see later that this particular converse is true, but to show why requires a separate argument.)

Set II (pages 69–70)

David Gale, Professor Emeritus of Mathematics at the University of California at Berkeley recalls the rotating pencil demonstration in exercises 38 through 40 as his first encounter with a mathematical proof. Another student showed it to him when he was in the fifth grade. (*Tracking The Automatic Ant*, by David Gale, Springer, 1998).

Eye Pupil.

•26. Approximately 50 mm². ($\pi 4^2 = 16\pi \approx 50$.)

27. Approximately 3 mm². ($\pi 1^2 = \pi \approx 3.14$.)

28. Approximately 16 times as much.

Triangle Angle Sum Theorem.

29. 90°.

30. 60°.

31. No, because then the third angle would have a measure of 0°.

•32. Yes, because the sum of the other two angles could be 1°.

33. Yes, because the sums of the angles in the two triangles are the same.

The Stretched Cord.

•34. a^2.

35. $2a^2$.

36. $2a^2$.

37. By the Pythagorean Theorem.

Pencil Experiment.

•38. To the left.

39. That the pencil has turned through 180°.

40. The sum of the angles of a triangle is 180°.

Aryabhata on the Circle.

41. Yes. $a = \frac{1}{2}c\frac{1}{2}d = \frac{1}{2}(2\pi r)\frac{1}{2}(2r) = \pi r^2$.

42. 3.1416. If the circumference is $8(4 + 100) + 62{,}000 = 62{,}832$ and the diameter is 20,000, then, because $c = \pi d$, $62{,}832 = \pi(20{,}000)$; so $\pi = \frac{62{,}832}{20{,}000} = 3.1416$.

43. Dilcue's answer is a good one because not only is PIE an old way to spell PI but his answer reflected in a vertical mirror is 3.14.

Set III (page 70)

Greg N. Frederickson, the author of *Dissections, Plane and Fancy* (Cambridge University Press, 1997), is a professor of computer science at Purdue University. His book is the definitive work on the amazing topic of geometric dissections and should be on the bookshelf of every geometry classroom.

When the three pieces are arranged as in the second figure, they seem to form a square whose side is the hypotenuse of one of the right triangles. When the pieces are arranged as in the fourth figure, they seem to form two squares whose sides are equal to the other two sides of the right triangle. The total area of the three pieces is the same no matter how they are arranged; so the two arrangements illustrate the Pythagorean Theorem.

Chapter 2, Review

Set I (pages 71–73)

A. Bartlett Giamatti was president of Yale University and, later, of the National Baseball League. He also served briefly as Commissioner of Major League Baseball until his death in 1989. *Take Time for Paradise* is a nice little book on sports and play.

The Morton Salt Company originally considered the slogan "Even in rainy weather, it flows freely" to promote the message that its salt would pour easily in damp weather. They then settled on a variation of the old proverb: "It never rains but it pours" for their slogan and have been using it on their packages since 1914.

The SAT problem was a multiple-choice question in its original format, and the problem asked only for *x*.

The judge in the trial excerpt was Judge Ito, and the conversation took place in Los Angeles on July 26, 1995, in the case of The People of the State of California vs. Orenthal James Simpson.

•1. If something is a limerick, then it has five lines.

2. If I perfect my perpetual motion machine, then I will make a fortune.

3. If something is a toadstool, then it is not edible.

4. $a \to b$, $b \to c$; therefore, $a \to c$.

5. If we have known freedom, then we fear its loss.

6. One or both of the premises being false.

7. If you are a daredevil, then you are recklessly bold.
If you are recklessly bold, then you are a daredevil.

8. Each is the converse of the other.

•9. Collinear.

10. Two.

•11. A postulate.

12. An Euler diagram.

•13. If it rains, then it pours.

14. It pours if it rains.

15. If it pours, then it rains.

16. No.

17. 100°. (180 – 48 – 32 = 100.)

•18. 50°. ($2x = 100$, $x = 50$.)

19. 98°. (180 – 50 – 32 = 98.)

•20. Three noncollinear points determine a plane.

21. If the feet of the four legs were not coplanar, the tripod might tip back and forth.

Geometry in Spanish.

22. In every right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

•23. Square.

24. Equal.

25. They are defined by other words, which results in going around in circles.

26. Point, line, plane.

27. Because we must have a starting point to avoid going around in circles. These words are especially simple and so serve as good starting points. (Note: The answer is *not* that we cannot define them.)

Courtroom Questions.

28. The witness thought π might be 2.12 or 2.17. The witness also said that pi times 2.5 squared is 19 but, rounded to the nearest whole number, the answer would be 20.

29. No. The judge guessed that pi was 3.1214, perhaps influenced by the witness's guess of 2.12, and 3.1214 is too small.

30. No.

Set II (pages 73–74)

In his book titled *Science Awakening* (Oxford University Press, 1961), B. L. van der Waerden said of the Egyptians' method for finding the area of a circle: "It is a great accomplishment of the Egyptians to have obtained such a good approximation. The Babylonians, who had reached a much higher stage of mathematical development, always used $\pi = 3$."

Although it is unlikely that your students will be surprised that the circumference/perimeter ratio of the circle to its circumscribed square is equal to the area/area ratio, this relation is remarkable. It holds true even if the circle and square are stretched into an ellipse and rectangle. John Pottage wrote a nice dialogue between Galileo's characters Simplicio and Segredo about this topic in the first chapter of his *Geometrical Investigations—Illustrating the Art of Discovery in the Mathematical Field* (Addison-Wesley, 1983).

The planet so small that someone can walk around it in 20 minutes is included in Lewis Carroll's *Sylvie and Bruno Concluded* (1893). More about Carroll's mathematical recreations can be found in Martin Gardner's *The Universe in a Handerchief* (Copernicus, 1996). Carroll wrote of the planet: "There had been a great battle . . . which had ended rather oddly: the vanquished army ran away at full speed, and in a very few minutes found themselves face-to-face with the victorious army, who were marching home again, and who were so frightened at finding themselves between *two* armies, that they surrendered at once!"

31. Postulates.

•32. Theorems.

33. Direct and indirect.

•34. 144 square units. ($12^2 = 144$.)

35. 256 square units. ($16^2 = 256$.)

36. 400 square units. ($144 + 256 = 400$.)

•37. 20 ft. ($x^2 = 16^2 + 12^2 = 400$, $x = \sqrt{400} = 20$.)

38. $(\frac{16}{9}r)^2 = (\frac{256}{81})r^2$; so π would be equal to $\frac{256}{81}$, or approximately 3.16.

39. They would be too large.

Panda Proof.

40. *Second statement:* If he had a sandwich, he would take out a gun.
Fourth statement: If he shot the waiter, he would leave without paying.

•41. Direct.

Clue Proof.

42. *Beginning assumption:* Suppose that Colonel Mustard didn't do it.
Missing statement: If Miss Scarlet did it, then it was done in the dining room.
Contradiction: It happened after 4 P.M.
Conclusion: Colonel Mustard did it.

43. Indirect.

Circle in the Square.

•44. $2\pi r$.

45. $8r$.

46. $\frac{2\pi r}{8r} = \frac{\pi}{4}$. ($\frac{\pi}{4}$ is approximately 0.785.)

47. πr^2.

•48. $4r^2$.

49. $\frac{\pi r^2}{4r^2} = \frac{\pi}{4}$.

50. Approximately 1,680 feet. If someone can walk three miles in an hour, they can walk one mile in 20 minutes. Because $c = \pi d$,

$$d = \frac{c}{\pi} = \frac{5{,}280}{\pi} \approx 1{,}680 \text{ feet.}$$

Chapter 2, Algebra Review (page 76)

•1. $10x$.

•2. $16x^2$.

•3. $6x + 5$.

•4. $4x - 5$.

•5. $6x^2$.

•6. $-6x^2$.

•7. $9x + 4y$.

•8. $3x + 2y$.

•9. $5x + 15$.

•10. $44 - 11x$.

•11. $18x + 6$.

•12. $40 - 56x$.

•13. $9x^2 - 18x$.

•14. $20x + 6x^2$.

•15. $x^2y + xy^2$.

•16. $2x^2 - 4x + 8$.

•17. $5x - 3 = 47$,
$5x = 50$,
$x = 10$.

•18. $9 + 2x = 25$,
$2x = 16$,
$x = 8$.

•19. $4x + 7x = 33$,
$11x = 33$,
$x = 3$.

•20. $10x = x + 54$,
$9x = 54$,
$x = 6$.

•21. $6x - 1 = 5x + 12$,
$x - 1 = 12$,
$x = 13$.

•22. $2x + 9 = 7x - 36$,
$-5x + 9 = -36$,
$-5x = -45$,
$x = 9$.

•23. $8 - x = x + 22$,
$8 - 2x = 22$,
$-2x = 14$,
$x = -7$.

•24. $3x - 5 = 10x + 30$,
$-7x - 5 = 30$,
$-7x = 35$,
$x = -5$.

•25. $4(x - 11) = 3x + 16$,
$4x - 44 = 3x + 16$,
$x - 44 = 16$,
$x = 60$.

•26. $x(x + 7) = x^2$,
$x^2 + 7x = x^2$,
$7x = 0$,
$x = 0$.

•27. $5(x + 2) = 2(x - 13)$,
$5x + 10 = 2x - 26$,
$3x + 10 = -26$,
$3x = -36$,
$x = -12$.

•28. $6(4x - 1) = 7(15 + 3x)$,
$24x - 6 = 105 + 21x$,
$3x - 6 = 105$,
$3x = 111$,
$x = 37$.

•29. $8 + 2(x + 3) = 10$,
$8 + 2x + 6 = 10$,
$2x + 14 = 10$,
$2x = -4$,
$x = -2$.

•30. $x + 7(x - 5) = 2(5 - x)$,
$x + 7x - 35 = 10 - 2x$,
$8x - 35 = 10 - 2x$,
$10x - 35 = 10$,
$10x = 45$,
$x = 4.5$.

•31. $4(x + 9) + x(x - 1) = x(6 + x)$,
$4x + 36 + x^2 - x = 6x + x^2$,
$3x + 36 = 6x$,
$36 = 3x$,
$x = 12$.

•32. $2x(x + 3) + x(x + 4) = 5x(x + 2) - 2x^2$,
$2x^2 + 6x + x^2 + 4x = 5x^2 + 10x - 2x^2$,
$3x^2 + 10x = 3x^2 + 10x$,
$0 = 0$;
so x can be any number.

Chapter 3, Lesson 1

Set I (pages 80–81)

Zero among all numbers is surely the cause of the most confusion. Ask anyone other than someone knowledgeable in mathematics what number one divided by zero is or what zero divided by zero is, and you will probably not get a meaningful answer. The problem, of course, is that there is no number equal to $\frac{n}{0}$ if $n \neq 0$, whereas quantities that approach $\frac{0}{0}$ may yield any number in the limit, a fact of which beginning calculus students become aware when they first encounter l'Hôpital's Rule. That is why $\frac{0}{0}$ itself must remain undefined as well.

Charles Seife remarks in his book titled *Zero—The Biography of a Dangerous Idea* (Viking, 2000): "If you wantonly divide by zero, you can destroy the entire foundation of logic and mathematics. Dividing by zero once—just one time—allows you to prove, mathematically, anything at all in the universe."

•1. Subtraction.

•2. Substitution.

3. Multiplication.

4. Substitution.

5. Division.

6. $a + b + c + d = b + e + f + g$.

7. Subtraction.

8. Subtraction.

9. Substitution.

•10. Addition.

11. Substitution.

12. Division (or multiplication by $\frac{1}{2}$).

13. 2.

•14. No number (because there is no number when multiplied by 0 that gives 6.)

15. 0.

16. Any number!

17. Any number can replace ?, although this fact makes the first equation seem strange for numbers other than 0 or 1.

18. Substitution.

•19. If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

If $c = d$, then $\frac{a}{c} = \frac{b}{d}$ by substitution.

Correcting Mistakes.

20. In the first step, 7 was subtracted from the left side of the equation but added to the right side. The equation should be $2x = 12$.

•21. Substitution.

22. Indirect.

Set II (pages 81–82)

We will return to the "sum of the area of the crescents equal to the area of the triangle" problem later in the course.

The figure from the Chinese text illustrating the Pythagorean Theorem is, of course, the one from Euclid's *Elements*. Although the Chinese text dates back only about 400 years, the Pythagorean Theorem may have been known in China long before Pythagoras was born. The theorem appears in the oldest Chinese mathematics book known, the *Chou Pei*, possibly written as long ago as 1100 B.C.

Triangle and Crescents.

23. They are equal.

24. By subtracting II + IV from both sides of the equation I + II + IV + V = II + III + IV, we get I + V = III.

Binomial Square.

•25. a^2 and b^2.

•26. $2ab$.

27. $a + b$.

•28. $(a + b)^2$.

29. $a^2 + 2ab + b^2$.

30. $(a + b)^2 = a^2 + 2ab + b^2$.

Chinese Proof.

31. The Pythagorean Theorem.

•32. Substitution.

33. Substitution.

34. Addition.

35. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

Circle Formula.

36. $a = \pi r^2$ and $r = \frac{d}{2}$; so $a = \pi(\frac{d}{2})^2 = \frac{\pi}{4}d^2$.

Quadratic Formula.

37. $ax^2 + bx + c = 0$.

•38. Division.

39. Subtraction.

40. Addition.

•41. A direct proof.

42. Substitution.

43. $x = \frac{-5 \pm \sqrt{25 - 24}}{4}$, $x = \frac{-5 \pm 1}{4}$;

$x = \frac{-5 + 1}{4} = \frac{-4}{4} = -1$;

$x = \frac{-5 - 1}{4} = \frac{-6}{4} = -1.5$.

Dilcue's Pie.

44. The area and circumference of the pie were evidently the same number.

45. If $\pi r^2 = 2\pi r$, then $\frac{\pi r^2}{\pi r} = \frac{2\pi r}{\pi r}$, so $r = 2$. The radius of the pie must have been 2 feet. The circumference must have been $2\pi(2) = 4\pi$ (or approximately 12.6) feet.

46. 4π (or approximately 12.6) *square* feet.

Set III (page 83)

There are quite a number of fallacious algebraic "proofs" based on dividing by zero in a disguised form. Carroll's version, accompanied by his charming whimsy, is undoubtedly the best—especially because it results in such an unexpected answer to one of the first multiplication problems that every child encounters.

Because $x = y$, $x - y = 0$. In dividing each side of the equation by $(x - y)$, we are dividing by 0, so the rest does not follow. This is especially obvious if we substitute 0 for $(x - y)$ in the equation $2(x + y)(x - y) = 5(x - y)$. It is certainly true that $2 \times 2 \times 0 = 5 \times 0$, but not that $2 \times 2 = 5$.

Chapter 3, Lesson 2

Set I (pages 86–88)

Although the triple jump was included in the first modern Olympics in 1896, it was probably not part of the ancient Greek Olympics. The triple jump is now thought to have originated as a result of a misunderstanding. The Greeks added the lengths of the three best jumps in the long-jump competition, which may have led to the idea that they were performing a "triple" jump. The world record for the triple jump is currently a little more than 18 meters.

The closest footwear that we currently have to "seven league boots" was reported in the July 15, 2000, issue of *New Scientist* magazine. A team of Russian engineers has developed fuel-powered boots that enable the wearer to jump over 6-foot walls, take 12-foot steps, and walk at speeds of 35 miles an hour!

Hotels and apartment buildings commonly have floor numbers that skip from 12 to 14, owing mainly to the possible superstition of potential residents. Even room numbers ending in 13 are often omitted.

1. The points on a line can be numbered so that positive number differences measure distances.

2. A point is between two other points on the same line iff its coordinate is between their coordinates.

3. If A-B-C, then AB + BC = AC.

•4. The definition of betweenness of points.

5. The Betweenness of Points Theorem.

Triple Jump.

6. 18 meters. (58 − 40 = 18.)

•7. 52. (47 + 5 = 52.)

8. 6 meters. (58 − 52 = 6.)

•9. BC + CD = BD, or 5 + 6 = 11.

10. AB + BD = AD, or 7 + 11 = 18.

11. AC + CD = AD, or 12 + 6 = 18.

Eclipse Problems.

12. Collinear.

13. Figure B.

14. 93,240,000 mi. (SE + EM = SM, 93,000,000 + 240,000 = 93,240,000.)

•15. 92,760,000 mi. (SM + ME = SE, SM + 240,000 = 93,000,000, SM = 92,760,000.)

Seven League Boots.

16. B, 11; C, 18; D, 25; E, 32; F, 39.

•17. 18.

18. 39.

19. How far (in leagues) the person is from home.

Skyscraper.

•20. 9 – 2 = 7.

21. 15 – 5 = 10.

22. No. The second answer is wrong because the ruler omits 13. (The skyscraper does not have a "13th" floor.)

Football Field.

23. A, 25; B, 35.

24. AB = 35 – 25 = 10.

25. The lines are not numbered like a ruler because to most numbers there correspond two points, not one.

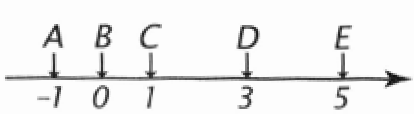
•26. A, –25; B, 15.

27. AB = 15 – (–25) = 40.

28. Yes.

SAT Problem.

29.



•30. B.

31. 6 units. [5 – (–1) = 6.]

Set II (pages 88–90)

When we observe stars at night, any three stars that seem to be collinear almost certainly are not. Such stars in space generally determine the vertices of a triangle that we happen to be seeing "on edge."

At first glance, the arrangement of the numbers on the yardstick in exercise 48 does not seem especially remarkable. The story behind them, however, is rather intriguing. Martin Gardner in his book titled *The Magic Numbers of Dr. Matrix* (Prometheus Books, 1985) noted: "Henry E. Dudeney, in problem 180, *Modern Puzzles* (1926), asked for the placing of eight marks on a 33-inch ruler and gave 16 solutions. Dudeney undoubtedly believed that at least nine marks were required for any ruler longer than 33 inches." Gardner reported that the yardstick problem of exercise 46 was solved by John Leech in 1956 and first published in a paper titled "On the Representation of 1, 2, . . . , *n* Differences" in the *Journal of the London Mathematical Society*. It has since been proved that Leech's solution for marking the yardstick with only 8 marks is the only one possible.

Pole Vaulter.

32. A point is between two other points on the same line iff its coordinate is between their coordinates.

33. The pole is bent; so the points are not collinear.

34. If A-B-C, then AB + BC = AC.

•35. AB + BC is greater than AC.

36. No.

Ladder Problem 1.

37. *(Student answer.)* ("Yes" seems reasonable.)

•38. AC = AB + BC.

39. BD = BC + CD.

40. Substitution.

41. Subtraction.

Ladder Problem 2.

42. B, 23; C, 35; D, 47; E, 59; F, 71; G, 83; H, 95; I, 107; Y, 118.

43. 107 – 11 = 96 and 8 × 12 = 96.

44. $2 \times 11 + 8 \times 12 = 22 + 96 = 118$.

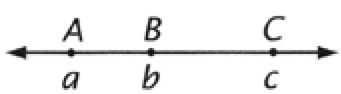
Stars Proof.

45. Proof.
Suppose that star Y is between star X and star Z.
If X-Y-Z, then XY + YZ = XZ.
If XY + YZ = XZ, then 7.2 + 9.8 = 16.6.
This contradicts the fact that $17.0 \neq 16.6$.
Therefore, our assumption is false and
star Y is not between star X and star Z.

Signal Problem.

46. If you don't change your speed, you will travel $3 \times 44 = 132$ feet in 3 seconds. Because A-B-C, AB + BC = AC; so $x + 60 = 132$ and $x = 72$. You can be 72 feet from the intersection.

47.



Proof.
(1) A-B-C. (The hypothesis.)
(2) $a > b > c$. (Definition of betweenness of points and this case.)
(3) AB = $a - b$ and BC = $b - c$. (Ruler Postulate.)
(4) AB + BC = $(a - b) + (b - c) = a - c$. (Addition.)
(5) AC = $a - c$. (Ruler Postulate.)
(6) AB + BC = AC. (Substitution.)

Yardstick Distances.

48.

1	AB, IJ	19	BF
2	BC	20	AF
3	AC, CD	21	DG
4	GH, HI	22	EI
5	BD, HJ	23	EJ
6	AD	24	CG
7	DE, EF, FG	25	DH
8	GI	26	BG
9	GJ	27	AG
10	CE	28	CH
11	FH	29	DI
12	BE	30	BH, DJ
13	AE	31	AH
14	DF, EG	32	CI
15	FI	33	CJ
16	FJ	34	BI
17	CF	35	AI, BJ
18	EH	36	AJ

Set III (page 90)

It is always fun to see good mathematics problems posed and solved in unexpected places. The oar problem is adapted from an example in the entry on "Oars" in John Lord's fascinating book titled *Sizes—The Illustrated Encyclopedia* (HarperCollins, 1995).

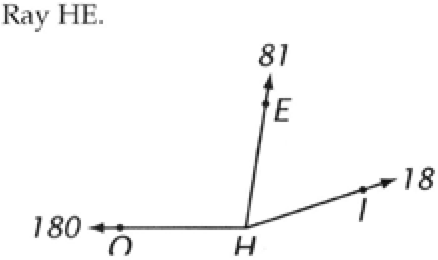
1. Half of CB is 21 inches and half of the overlap is 2 inches; so AB and CD should each be 21 + 2 = 23 inches.
2. Because AB = $7x = 23$, $x = \frac{23}{7}$. Because AF = $25x$, AF = $25(\frac{23}{7}) \approx 82$ inches. (Notice that the length of the paddle is not needed.)
3. Because 82 inches ≈ 6.8 feet, you should probably order 7-foot oars.

Chapter 3, Lesson 3

Set I (pages 93–95)

More examples of rotations in sports include somersaults in gymnastics and spins in figure skating. Single, double, and triple axels in skating are named for their inventor, Norwegian skater Axel Paulsen.

1. The rays in a half-rotation can be numbered from 0 to 180 so that positive number differences measure angles.
2. A ray is between two others in the same half-rotation iff its coordinate is between their coordinates.
3. If OA-OB-OC, then ∠AOB + ∠BOC = ∠AOC.
4. Ray HE.



5. ∠OHE = 99°, ∠EHI = 63°, and ∠OHI = 162°.
•6. The definition of betweenness of rays.
7. The Protractor Postulate.

•8. Rays.

9. Its sides.

10. ∠A, ∠CAB, ∠PAC.

Danger Area.

•11. The coordinates.

12. 75°.

•13. Acute.

14. One revolution. (The player jumps, spins around once, and releases the ball before landing.)

15. The diver turns one and a half rotations, or 540°, during the dive.

•16. 15. (One each at A, E, and G, and three each at B, C, D, and F.)

17. All 15 of them.

18. ∠CBG, ∠CDE, ∠CFE, ∠CFG.

19. ∠ABG, ∠BCF, ∠DCF, ∠ADE, ∠E, ∠G.

20. No. We do not know from what perspective the photograph was taken.

Bubble Angles.

21. Right.

•22. Obtuse.

23. 90°. ($\frac{360}{4} = 90$.)

24. 120°. ($\frac{360}{3} = 120$.)

Cactus Spokes.

•25. 14.4°. (There are 25 spokes and hence 25 such angles; $\frac{360}{25} = 14.4$.)

•26. Yes.

27. Its coordinate is between their coordinates. (CS-CP-CK because 43.2 < 57.6 < 86.4.)

28. 14.4°. (57.6 – 43.2 = 14.4.)

•29. 28.8°. (86.4 – 57.6 = 28.8.)

30. 43.2°. (86.4 – 43.2 = 43.2.)

•31. ∠SCP + ∠PCK = ∠SCK.

32. The Betweenness of Rays Theorem.

33. If OA-OB-OC, then ∠AOB + ∠BOC = ∠AOC.

Set II (pages 95–97)

Runway Numbers.

34. 18.

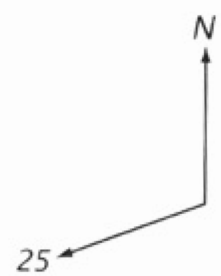
•35. 27.

36. They are $\frac{1}{10}$th of the numbers on the protractor.

37. 23. (50° + 180° = 230°.)

•38. They are the same.

39. 110°. (360° – 250°.)



40. The left runway and the right runway.

Pool Ball Angles.

•41. Betweenness of Rays Theorem. (If OA-OB-OC, then ∠AOB + ∠BOC = ∠AOC.)

42. Substitution.

43. Subtraction.

NBC Peacock.

•44. 20°. ($\frac{180}{9} = 20$.)

45. 18°. ($\frac{180}{10} = 18$.)

46. 15°. ($\frac{180}{12} = 15$.)

47. 16.363636 . . .° (or $16\frac{4}{11}$°). ($\frac{180}{11} = 16\frac{4}{11}$.)

Minutes and Seconds.

48. An hour.

49. 3,600. (1° = 60' = 60 × 60" = 3,600".)

•50. 360.

•51. 15. $\left(\frac{360°}{24 \text{ hours}}.\right)$

52. 15.

53. 15.

54. *Proof.*
    (1) OA-OB-OC. (The hypothesis.)
    (2) $a < b < c$. (Definition of betweenness of rays and this case.)
    (3) $\angle AOB = b - a$ and $\angle BOC = c - b$. (Protractor Postulate.)
    (4) $\angle AOB + \angle BOC = (b - a) + (c - b) = c - a$. (Addition.)
    (5) $\angle AOC = c - a$. (Protractor Postulate.)
    (6) $\angle AOB + \angle BOC = \angle AOC$. (Substitution.)

Set III (page 97)

This exercise is another example of the need for deductive reasoning and proof. No matter how accurately the figure may be drawn, measurements of ∠EFB can neither give its exact value nor reveal why it should be 20°. J. L. Heilbron calls it "The Tantalus problem" after Tantalus, the character in Greek mythology from which the word "tantalize" is derived. The name is fitting because the prospective solver of the problem is "tormented with something desired but out of reach."

A nice discussion of the problem and a similar but easier one are included on pages 292–295 of Heilbron's *Geometry Civilized* (Clarendon Press, 1998). More interesting information about the Tantalus problem is included in *Tracking the Automatic Ant*, by David Gale (Springer, 1998). See pages 123–125 on "Configurations with Rational Angles" and the research done on the problem by Armando Machado on pages 227–232.

1. *(Student drawing of figure.)*

2. ∠AGB = 50°, ∠AGF = 130°, ∠BGF = 130°, ∠C = 20°.

3. On an accurate drawing, it should appear to be about 20°.

Chapter 3, Lesson 4

Set I (pages 100–101)

The arrow figure in exercises 12 through 14 is a variation of the "Muller-Lyer illusion." It was discovered in 1899 by C. H. Judd, who mentioned it in an article on geometrical illusions published in *Psychological Review*. Other variations of this illusion are included in *Can You Believe Your Eyes?* by J. R. Block and Harold E. Yuker (Brunner/Mazel, 1992).

Origami is the subject of one of Martin Gardner's early "Mathematical Games" columns in *Scientific American* (July 1959), reprinted in *The 2nd Scientific American Book of Mathematical Puzzles and Diversions* (Simon & Schuster, 1961). As a child, Peter Engel was inspired by this column and has written a wonderful book on the subject, *Folding the Universe* (Vintage Books, 1989). Beyond including detailed procedures for folding a large variety of figures from simple to extremely complex, Engel relates the history of origami and its connections with many other fields. For example, in a section of the book titled "The Psychology of Invention," Engel includes a summary of French mathematician Jacques Hadamard's remarks on the subject of invention in mathematics. *Folding the Universe* is without doubt the definitive book on origami for the mathematics classroom.

1. Points that are on the same line.

2. Lines that contain the same point.

•3. Figures that can be made to coincide (or fit exactly together).

4. A drawing made with a straightedge and compass.

5. A statement formed by interchanging the hypothesis and conclusion of another statement.

6. A theorem that is easily proved as a consequence of another statement.

•7. Then it divides it into two equal segments.

8. If a point divides a line segment into two equal segments, then it is the midpoint of it.

•9. Each is the converse of the other.

10. A line segment has exactly one midpoint.

11. An angle has exactly one ray that bisects it.

Arrow Illusion.

12. (Most people would say M.)

13. No, because a line segment has exactly one midpoint.

14. AN = NB.

Origami Duck.

•15. No.

16. Yes.

17. ∠BAD.

18. ∠BCD, ∠ACD, and ∠BCA.

•19. 45°.

20. 22.5°.

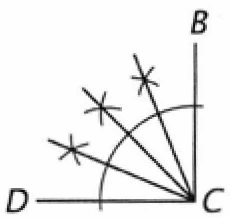
21. 45°.

•22. 67.5°.

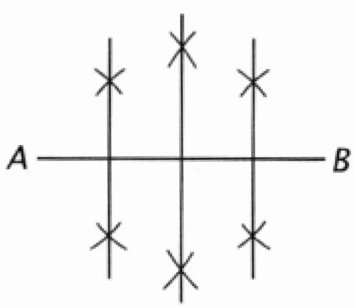
23. 67.5° (because in triangle DFC, ∠D = 90° and ∠FCD = 22.5°).

Bisection Constructions.

24.



25.



Clock Puzzle.

•26. The midpoint of a line segment divides it into two equal segments.

27. 1.5.

•28. 1.5.

29. D, 4.5; E, 6; F, 7.5.

30. 7.5 seconds.

Set II (pages 102–104)

Students taking chemistry might enjoy identifying the atoms in the acetylene molecule: two carbon atoms (points B and C in the figure) and two hydrogen atoms (points A and D). According to Linus Pauling, the H–C bond length is 1.1 angstroms and the C–C bond length is 1.2 angstroms.

Euclid defined an angle as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line"; so, for Euclid, straight angles are not considered.

David W. Henderson, in his valuable book titled *Experiencing Geometry* (Prentice Hall, 2001), offers some important insights with regard to the question, "What is an angle?" He writes:

"I looked in all the plane geometry books in the university library and found their definitions for 'angle.' I found nine different definitions! Each expressed a different meaning or aspect of 'angle' . . . .

There are at least three different perspectives from which one can define 'angle,' as follows:

a *dynamic* notion of angle—angle as *movement*;

angle as *measure*; and,

angle as *geometric shape* . . . .

Each of these perspectives carries with it methods for checking angle congruency. You can check the congruency of two dynamic angles by verifying that the actions involved in creating or replicating them are the same. If you feel that an angle is a measure, then you must verify that both angles have the same measure. If you describe angles as geometric shapes, then one angle should be made to coincide with the other using isometries in order to prove angle congruence."

Yen Measurement.

•31. 2 cm.

•32. 2π (or about 6.28). ($c = \pi d = 2\pi$.)

33. π (or about 3.14).

Acetylene Proofs.

34. Addition.
Betweenness of Points Theorem.
Substitution.

35. Suppose that B is the midpoint of AC.
The midpoint of a line segment divides it into two equal segments.
Substitution.
The fact that AC > 2AB.
B is not the midpoint of AC.

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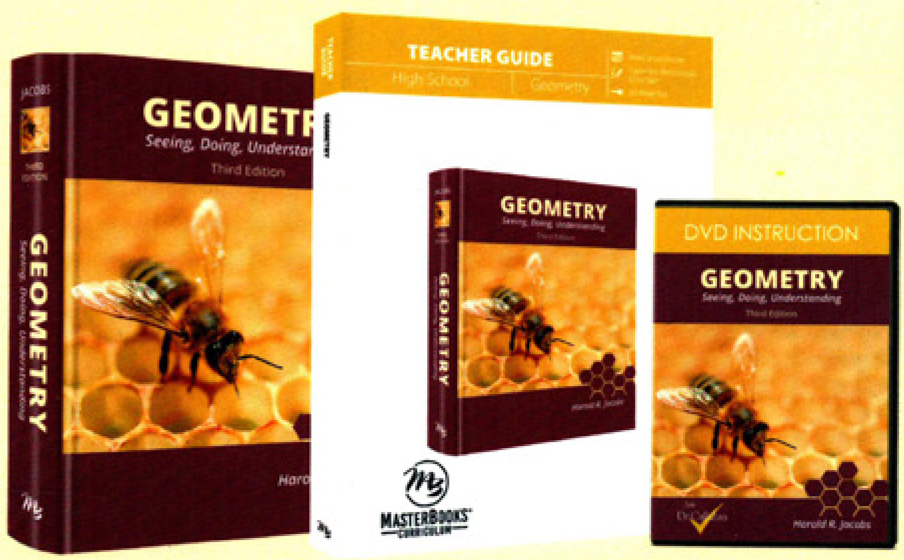

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