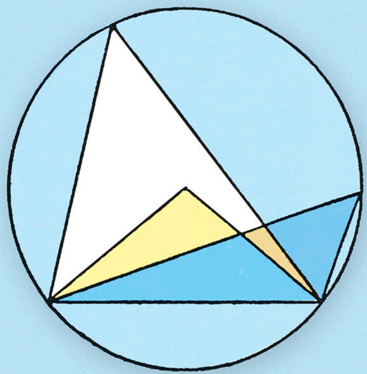
Vol.1 (Books I and II)

EUCLID

THE THIRTEEN BOOKS OF THE ELEMENTS

Translated with introduction and commentary by Sir Thomas L. Heath



Second Edition Unabridged

THE THIRTEEN BOOKS
OF
EUCLID'S ELEMENTS

τοῦ ἰησοῦ. οὐκ ἔστη
[illegible] ἀξιομειων
[illegible] ἀλλὰ
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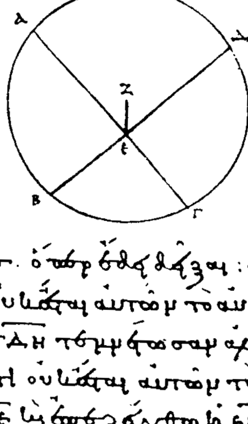
ε

Ἐὰν ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας μὴ διὰ τοῦ κέντρου οὖσαι, οὐ τέμνουσιν ἀλλήλας δίχα. ἔστω κύκλος ὁ ΑΒΓΔ καὶ ἐν αὐτῷ δύο εὐθεῖαι αἱ ΑΓ ΒΔ τεμνέτωσαν ἀλλήλας κατὰ τὸ Ε μὴ διὰ τοῦ κέντρου οὖσαι. λέγω ὅτι οὐ τέμνουσιν ἀλλήλας δίχα. εἰ γὰρ δυνατόν, τεμνέτωσαν ἀλλήλας δίχα ὥστε ἴσην εἶναι τὴν μὲν ΑΕ τῇ ΕΓ, τὴν δὲ ΒΕ τῇ ΕΔ. καὶ εἰλήφθω τὸ κέντρον τοῦ ΑΒΓΔ κύκλου καὶ ἔστω τὸ Ζ καὶ ἐπεζεύχθω ἡ ΖΕ. ἐπεὶ οὖν εὐθεῖά τις διὰ τοῦ κέντρου ἡ ΖΕ εὐθεῖάν τινα τὴν ΑΓ δίχα τέμνει, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει. ὀρθὴ ἄρα ἐστὶν ἡ ὑπὸ ΖΕΑ. πάλιν ἐπεὶ εὐθεῖά τις ἡ ΖΕ εὐθεῖαν τινὰ τὴν ΒΔ δίχα τέμνει, καὶ πρὸς ὀρθὰς αὐτὴν τέμνει. ὀρθὴ ἄρα ἡ ὑπὸ ΖΕΒ. ἐδείχθη δὲ καὶ ἡ ὑπὸ ΖΕΑ ὀρθή. ἴση ἄρα ἡ ὑπὸ ΖΕΑ τῇ ὑπὸ ΖΕΒ, ἡ ἐλάττων τῇ μείζονι. ὅπερ ἐστὶν ἀδύνατον. οὐκ ἄρα αἱ ΑΓ ΒΔ τέμνουσιν ἀλλήλας δίχα. ἐὰν ἄρα ἐν κύκλῳ δύο εὐθεῖαι τέμνωσιν ἀλλήλας καὶ τὰ ἑξῆς. ὅπερ ἔδει δεῖξαι:-



Ἐὰν δύο κύκλοι τέμνωσιν ἀλλήλους, οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον. δύο γὰρ κύκλοι οἱ ΑΒΓ ΓΔΗ τεμνέτωσαν ἀλλήλους κατὰ τὰ Β Γ σημεῖα. λέγω ὅτι οὐκ ἔσται αὐτῶν τὸ αὐτὸ κέντρον. εἰ γὰρ δυνατόν, ἔστω τὸ Ε καὶ ἐπεζεύχθω ἡ ΕΓ καὶ διήχθω ἡ ΕΖΗ ὡς ἔτυχεν. καὶ ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΑΒΓ κύκλου, ἴση ἐστὶν ἡ ΕΓ τῇ ΕΖ. πάλιν ἐπεὶ τὸ Ε σημεῖον κέντρον ἐστὶ τοῦ ΓΔΗ κύκλου, ἴση ἐστὶν ἡ ΕΓ τῇ ΕΗ. ἐδείχθη δὲ ἡ ΕΓ καὶ τῇ ΕΖ ἴση. καὶ ἡ ΕΖ ἄρα τῇ ΕΗ

THE THIRTEEN BOOKS OF EUCLID'S ELEMENTS

TRANSLATED FROM THE TEXT OF HEIBERG

WITH INTRODUCTION AND COMMENTARY

BY

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VOLUME I

INTRODUCTION AND BOOKS I, II

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PREFACE

"THERE never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures (we do not speak of *corrections* or *extensions* or *developments*) from the plan laid down by Euclid." De Morgan wrote thus in October 1848 (*Short supplementary remarks on the first six Books of Euclid's Elements* in the *Companion to the Almanac* for 1849); and I do not think that, if he had been living to-day, he would have seen reason to revise the opinion so deliberately pronounced sixty years ago. It is true that in the interval much valuable work has been done on the continent in the investigation of the first principles, including the formulation and classification of axioms or postulates which are necessary to make good the deficiencies of Euclid's own explicit postulates and axioms and to justify the further assumptions which he tacitly makes in certain propositions, content apparently to let their truth be inferred from observation of the figures as drawn; but, once the first principles are disposed of, the body of doctrine contained in the recent text-books of elementary geometry does not, and from the nature of the case cannot, show any substantial differences from that set forth in the *Elements*. In England it would seem that far less of scientific value has been done; the efforts of a multitude of writers have rather been directed towards producing alternatives for Euclid which shall be more suitable, that is to say, easier, for schoolboys. It is of course not surprising that, in these days of short cuts, there should have arisen a movement to get rid of Euclid and to substitute a "royal road to geometry"; the marvel is that a book which was not written for schoolboys but for grown men (as all internal evidence shows, and in particular the essentially theoretical character of the work and its aloofness from anything of the nature of "practical" geometry) should have held its own as a school-book for so long. And now that Euclid's proofs and arrangement are no longer required from candidates at examinations there has been a rush of competitors anxious to be first in the field with a new text-book on the more "practical" lines which now find so much favour. The natural desire of each teacher who writes such a text-book is to give prominence to some special nostrum which he has found successful with pupils. One result is, too often, a loss of a due sense of proportion; and, in any case, it is inevitable that there should be great diversity of treatment. It was with reference to such a danger that Lardner wrote in 1846: "Euclid once superseded, every teacher would esteem his own work the best, and every school would have its own class book. All that rigour and exactitude which have so long excited the admiration of men of science would be at an end. These very words would lose all definite meaning. Every school would have a different standard; matter of assumption in one being matter of demonstration in another; until, at length, GEOMETRY, in the ancient sense of the word, would be altogether frittered away or be only considered as a particular application of Arithmetic and Algebra." It is, perhaps, too early yet to prophesy what will be the ultimate outcome of the new order of things; but it would at least seem possible that history will repeat itself and that, when chaos has come again in geometrical teaching, there will be a return to Euclid more or less complete for the purpose of standardising it once more.

But the case for a new edition of Euclid is independent of any controversies as to how geometry shall be taught to schoolboys. Euclid's work will live long after all the text-books of the present day are superseded and forgotten. It is one of the noblest monuments of antiquity; no mathematician worthy of the name can afford not to know Euclid, the real Euclid as distinct from any revised or rewritten versions which will serve for schoolboys or engineers. And, to know Euclid, it is necessary to know his language, and, so far as it can be traced, the history of the "elements" which he collected in his immortal work.

This brings me to the *raison d'être* of the present edition. A new translation from the Greek was necessary for two reasons. First, though some time has elapsed since the appearance of Heiberg's definitive text and prolegomena, published between 1883 and 1888, there has not been, so far as I know, any attempt to make a faithful translation from it into English even of the Books which are commonly read. And, secondly, the other Books, VII. to X. and XIII., were not included by Simson and the editors who followed him, or apparently in any English translation since Williamson's (1781—8), so that they are now practically inaccessible to English readers in any form.

In the matter of notes, the edition of the first six Books in Greek and Latin with notes by Camerer and Haúber (Berlin, 1824—5) is a perfect mine of information. It would have been practically impossible to make the notes more exhaustive at the time when they were written. But the researches of the last thirty or forty years into the history of mathematics (I need only mention such names as those of Bretschneider, Hankel, Moritz Cantor, Hultsch, Paul Tannery, Zeuthen, Loria, and Heiberg) have put the whole subject upon a different plane. I have endeavoured in this edition to take account of all the main results of these researches up to the present date. Thus, so far as the geometrical Books are concerned, my notes are intended to form a sort of dictionary of the history of elementary geometry, arranged according to subjects; while the notes on the arithmetical Books VII.—IX. and on Book X. follow the same plan.

I desire to express here my thanks to my brother, Dr R. S. Heath, Vice-Principal of Birmingham University, for suggestions on the proof sheets and, in particular, for the reference to the parallelism between Euclid's definition of proportion and Dedekind's theory of irrationals, to Mr R. D. Hicks for advice on a number of difficult points of translation, to Professor A. A. Bevan for help in the transliteration of Arabic names, and to the Curators and Librarian of the Bodleian Library for permission to reproduce, as frontispiece, a page from the famous Bodleian MS. of the *Elements*. Lastly, my best acknowledgments are due to the Syndics of the Cambridge University Press for their ready acceptance of the work, and for the zealous and efficient coöperation of their staff which has much lightened the labour of seeing the book through the Press.

November, 1908.

PREFACE TO THE SECOND EDITION

I LIKE to think that the exhaustion of the first edition of this work furnishes a new proof (if such were needed) that Euclid is far from being defunct or even dormant, and that, so long as mathematics is studied, mathematicians will find it necessary and worth while to come back again and again, for one purpose or another, to the twenty-two-centuries-old book which, notwithstanding its imperfections, remains the greatest elementary textbook in mathematics that the world is privileged to possess.

The present edition has been carefully revised throughout, and a number of passages (sometimes whole pages) have been rewritten, with a view to bringing it up to date. Some not inconsiderable additions have also been made, especially in the Excursuses to Volume I, which will, I hope, find interested readers.

Since the date of the first edition little has happened in the domain of geometrical teaching which needs to be chronicled. Two distinct movements however call for notice.

The first is a movement having for its object the mitigation of the difficulties (affecting in different ways students, teachers and examiners) which are found to arise from the multiplicity of the different textbooks and varying systems now in use for the teaching of elementary geometry. These difficulties have evoked a widespread desire among teachers for the establishment of an agreed sequence to be generally adopted in teaching the subject. One proposal to this end has already been made: but the chance of the acceptance of an agreed sequence has in the meantime been prejudiced by a second movement which has arisen in other quarters.

I refer to the movement in favour of reviving, in a modified form, the proposal made by Wallis in 1663 to replace Euclid's Parallel-Postulate by a Postulate of Similarity (as to which see pp. 210—11 of Volume I of this work). The form of Postulate now suggested is an assumption that "Given one triangle, there can be constructed, on any arbitrary base, another triangle equiangular with (or similar to) the given triangle." It may perhaps be held that this assumption has the advantage of not referring, in the statement of it, to the fact that a straight line is of unlimited length; but, on the other hand, as is well known, Saccheri showed (1733) that it involves more than is necessary to enable Euclid's Postulate to be proved. In any case it would seem certain that a scheme based upon the proposed Postulate, if made scientifically sound, must be more difficult than the procedure now generally followed. This being so, and having regard to the facts (1) that the difference between the suggested Postulate and that of Euclid is in effect so slight and (2) that the historic interest of Euclid's Postulate is so great, I am of opinion that the proposal is very much to be deprecated.

T. L. H.

December 1925.

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INTRODUCTION.

THE *ELEMENTS*.

INTRODUCTION.

CHAPTER I.

EUCLID AND THE TRADITIONS ABOUT HIM.

As in the case of the other great mathematicians of Greece, so in Euclid's case, we have only the most meagre particulars of the life and personality of the man.

Most of what we have is contained in the passage of Proclus' summary relating to him, which is as follows[1]:

"Not much younger than these (sc. Hermotimus of Colophon and Philippus of Medma) is Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived[2] in the time of the first Ptolemy. For Archimedes, who came immediately after the first (Ptolemy)[3], makes mention of Euclid: and, further, they say that Ptolemy once asked him if there was in geometry any shorter way than that of the elements, and he answered that there was no royal road to geometry[4]. He is then younger than the pupils of Plato but older than Eratosthenes and Archimedes; for the latter were contemporary with one another, as Eratosthenes somewhere says."

This passage shows that even Proclus had no direct knowledge of Euclid's birthplace or of the date of his birth or death. He proceeds by inference. Since Archimedes lived just after the first

[1] Proclus, ed. Friedlein, p. 68, 6—20.

[2] The word γέγονε must apparently mean "flourished," as Heiberg understands it (*Litterargeschichtliche Studien über Euklid*, 1882, p. 26), not "was born," as Hankel took it: otherwise part of Proclus' argument would lose its cogency.

[3] So Heiberg understands ἐπιβαλὼν τῷ πρώτῳ (sc. Πτολεμαίῳ). Friedlein's text has καὶ between ἐπιβαλὼν and τῷ πρώτῳ; and it is right to remark that another reading is καὶ ἐν τῷ πρώτῳ (without ἐπιβαλών) which has been translated "in his first *book*," by which is understood *On the Sphere and Cylinder* I., where (1) in Prop. 2 are the words "let *BC* be made equal to *D by the second* (proposition) *of the first* of Euclid's (books)," and (2) in Prop. 6 the words "For these things are handed down in the Elements" (without the name of Euclid). Heiberg thinks the former passage is referred to, and that Proclus must therefore have had before him the words "by the second of the first of Euclid": a fair proof that they are genuine, though in themselves they would be somewhat suspicious.

[4] The same story is told in Stobaeus, *Ecl.* (II. p. 228, 30, ed. Wachsmuth) about Alexander and Menaechmus. Alexander is represented as having asked Menaechmus to teach him geometry concisely, but he replied: "O king, through the country there are royal roads and roads for common citizens, but in geometry there is one road for all."

Ptolemy, and Archimedes mentions Euclid, while there is an anecdote about *some* Ptolemy and Euclid, *therefore* Euclid lived in the time of the first Ptolemy.

We may infer then from Proclus that Euclid was intermediate between the first pupils of Plato and Archimedes. Now Plato died in 347/6, Archimedes lived 287–212, Eratosthenes *c.* 284–204 B.C. Thus Euclid must have flourished *c.* 300 B.C., which date agrees well with the fact that Ptolemy reigned from 306 to 283 B.C.

It is most probable that Euclid received his mathematical training in Athens from the pupils of Plato; for most of the geometers who could have taught him were of that school, and it was in Athens that the older writers of elements, and the other mathematicians on whose works Euclid's *Elements* depend, had lived and taught. He may himself have been a Platonist, but this does not follow from the statements of Proclus on the subject. Proclus says namely that he was of the school of Plato and in close touch with that philosophy[1]. But this was only an attempt of a New Platonist to connect Euclid with his philosophy, as is clear from the next words in the same sentence, "for which reason also he set before himself, as the end of the whole Elements, the construction of the so-called Platonic figures." It is evident that it was only an idea of Proclus' own to infer that Euclid was a Platonist because his *Elements* end with the investigation of the five regular solids, since a later passage shows him hard put to it to reconcile the view that the construction of the five regular solids was the end and aim of the *Elements* with the obvious fact that they were intended to supply a foundation for the study of geometry in general, "to make perfect the understanding of the learner in regard to the whole of geometry[2]." To get out of the difficulty he says[3] that, if one should ask him what was the aim (σκοπός) of the treatise, he would reply by making a distinction between Euclid's intentions (1) as regards the subjects with which his investigations are concerned, (2) as regards the learner, and would say as regards (1) that "the whole of the geometer's argument is concerned with the cosmic figures." This latter statement is obviously incorrect. It is true that Euclid's *Elements* end with the construction of the five regular solids; but the planimetrical portion has no direct relation to them, and the arithmetical no relation at all; the propositions about them are merely the conclusion of the stereometrical division of the work.

One thing is however certain, namely that Euclid taught, and founded a school, at Alexandria. This is clear from the remark of Pappus about Apollonius[4]: "he spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought."

It is in the same passage that Pappus makes a remark which might, to an unwary reader, seem to throw some light on the

[1] Proclus, p. 68, 20, *καὶ τῇ προαιρέσει δὲ* Πλατωνικός *ἐστι καὶ τῇ φιλοσοφίᾳ ταύτῃ οἰκεῖος.*

[2] *ibid.* p. 71, 8. [3] *ibid.* p. 70, 19 sqq.

[4] Pappus, VII. p. 678, 10—12, *συσχολάσας τοῖς ὑπὸ* Εὐκλείδου *μαθηταῖς ἐν* Ἀλεξανδρείᾳ *πλεῖστον χρόνον, ὅθεν ἔσχε καὶ τὴν τοιαύτην ἕξιν οὐκ ἀμαθῆ.*

personality of Euclid. He is speaking about Apollonius' preface to the first book of his *Conics*, where he says that Euclid had not completely worked out the synthesis of the "three- and four-line locus," which in fact was not possible without some theorems first discovered by himself. Pappus says on this[1]: "Now Euclid—regarding Aristaeus as deserving credit for the discoveries he had already made in conics, and without anticipating him or wishing to construct anew the same system (such was his scrupulous fairness and his exemplary kindliness towards all who could advance mathematical science to however small an extent), being moreover in no wise contentious and, though exact, yet no braggart like the other [Apollonius]—wrote so much about the locus as was possible by means of the conics of Aristaeus, without claiming completeness for his demonstrations." It is however evident, when the passage is examined in its context, that Pappus is not following any tradition in giving this account of Euclid: he was offended by the terms of Apollonius' reference to Euclid, which seemed to him unjust, and he drew a fancy picture of Euclid in order to show Apollonius in a relatively unfavourable light.

Another story is told of Euclid which one would like to believe true. According to Stobaeus[2], "some one who had begun to read geometry with Euclid, when he had learnt the first theorem, asked Euclid, 'But what shall I get by learning these things?' Euclid called his slave and said 'Give him threepence, since he must make gain out of what he learns.'"

In the middle ages most translators and editors spoke of Euclid as Euclid *of Megara*. This description arose out of a confusion between our Euclid and the philosopher Euclid of Megara who lived about 400 B.C. The first trace of this confusion appears in Valerius Maximus (in the time of Tiberius) who says[3] that Plato, on being appealed to for a solution of the problem of doubling the cubical altar, sent the inquirers to "Euclid the geometer." There is no doubt about the reading, although an early commentator on Valerius Maximus wanted to correct "Eucliden" into "*Eudoxum*," and this correction is clearly right. But, if Valerius Maximus took Euclid the geometer for a contemporary of Plato, it could only be through confusing him with Euclid of Megara. The first specific reference to Euclid as Euclid of Megara belongs to the 14th century, occurring in the ὑπομνηματισμοί of Theodorus Metochita (d. 1332) who speaks of "Euclid of Megara, the Socratic philosopher, contemporary of Plato," as the author of treatises on plane and solid geometry, data, optics etc.: and a Paris MS. of the 14th century has "Euclidis philosophi Socratici liber elementorum." The misunderstanding was general in the period from Campanus' translation (Venice 1482) to those of Tartaglia (Venice 1565) and Candalla (Paris 1566). But one Constantinus Lascaris (d. about 1493) had already made the proper

[1] Pappus, VII. pp. 676, 25—678, 6. Hultsch, it is true, brackets the whole passage pp. 676, 25—678, 15, but apparently on the ground of the diction only.

[2] Stobaeus, *l.c.*

[3] VIII. 12, ext. 1.

distinction by saying of our Euclid that "he was different from him of Megara of whom Laertius wrote, and who wrote dialogues"[1]; and to Commandinus belongs the credit of being the first translator[2] to put the matter beyond doubt: "Let us then free a number of people from the error by which they have been induced to believe that our Euclid is the same as the philosopher of Megara" etc.

Another idea, that Euclid was born at Gela in Sicily, is due to the same confusion, being based on Diogenes Laertius' description[3] of the philosopher Euclid as being "of Megara, or, according to some, of Gela, as Alexander says in the Διαδοχαί."

In view of the poverty of Greek tradition on the subject even as early as the time of Proclus (410–485 A.D.), we must necessarily take *cum grano* the apparently circumstantial accounts of Euclid given by Arabian authors; and indeed the origin of their stories can be explained as the result (1) of the Arabian tendency to romance, and (2) of misunderstandings.

We read[4] that "Euclid, son of Naucrates, grandson of Zenarchus[5], called the author of geometry, a philosopher of somewhat ancient date, a Greek by nationality domiciled at Damascus, born at Tyre, most learned in the science of geometry, published a most excellent and most useful work entitled the foundation or elements of geometry, a subject in which no more general treatise existed before among the Greeks: nay, there was no one even of later date who did not walk in his footsteps and frankly profess his doctrine. Hence also Greek, Roman and Arabian geometers not a few, who undertook the task of illustrating this work, published commentaries, scholia, and notes upon it, and made an abridgment of the work itself. For this reason the Greek philosophers used to post up on the doors of their schools the well-known notice: 'Let no one come to our school, who has not first learned the elements of Euclid.'" The details at the beginning of this extract cannot be derived from Greek sources, for even Proclus did not know anything about Euclid's father, while it was not the Greek habit to record the names of grandfathers, as the Arabians commonly did. Damascus and Tyre were no doubt brought in to gratify a desire which the Arabians always showed to connect famous Greeks in some way or other with the East. Thus Naṣīraddīn, the translator of the *Elements*, who was of Ṭūs in Khurāsān, actually makes Euclid out to have been "Thusinus" also[6]. The readiness of the Arabians to run away with an idea is illustrated by the last words

[1] Letter to Fernandus Acuna, printed in Maurolycus, *Historia Siciliae*, fol. 21 r. (see Heiberg, *Euklid-Studien*, pp. 22—3, 25).

[2] Preface to translation (Pisauri, 1572).

[3] Diog. L. II. 106, p. 58 ed. Cobet.

[4] Casiri, *Bibliotheca Arabico-Hispana Escurialensis*, I. p. 339. Casiri's source is al-Qifṭī (d. 1248), the author of the *Ta'rīkh al-Ḥukamā*, a collection of biographies of philosophers, mathematicians, astronomers etc.

[5] The *Fihrist* says "son of Naucrates, the son of Berenice (?)" (see Suter's translation in *Abhandlungen zur Gesch. d. Math.* VI. Heft, 1892, p. 16).

[6] The same predilection made the Arabs describe Pythagoras as a pupil of the wise Salomo, Hipparchus as the exponent of Chaldaean philosophy or as the Chaldaean, Archimedes as an Egyptian etc. (Ḥājī Khalfa, *Lexicon Bibliographicum*, and Casiri).

of the extract. Everyone knows the story of Plato's inscription over the porch of the Academy: "let no one unversed in geometry enter my doors"; the Arab turned geometry into *Euclid's* geometry, and told the story of Greek philosophers in general and "*their* Academies."

Equally remarkable are the Arabian accounts of the relation of Euclid and Apollonius[1]. According to them the *Elements* were originally written, not by Euclid, but by a man whose name was Apollonius, a carpenter, who wrote the work in 15 books or sections[2]. In the course of time some of the work was lost and the rest became disarranged, so that one of the kings at Alexandria who desired to study geometry and to master this treatise in particular first questioned about it certain learned men who visited him and then sent for Euclid who was at that time famous as a geometer, and asked him to revise and complete the work and reduce it to order. Euclid then re-wrote it in 13 books which were thereafter known by his name. (According to another version Euclid composed the 13 books out of commentaries which he had published on two books of Apollonius on conics and out of introductory matter added to the doctrine of the five regular solids.) To the thirteen books were added two more books, the work of others (though some attribute these also to Euclid) which contain several things not mentioned by Apollonius. According to another version Hypsicles, a pupil of Euclid at Alexandria, offered to the king and published Books XIV. and XV., it being also stated that Hypsicles had "discovered" the books, by which it appears to be suggested that Hypsicles had edited them from materials left by Euclid.

We observe here the correct statement that Books XIV. and XV. were not written by Euclid, but along with it the incorrect information that Hypsicles, the author of Book XIV., wrote Book XV. also.

The whole of the fable about Apollonius having preceded Euclid and having written the *Elements* appears to have been evolved out of the preface to Book XIV. by Hypsicles, and in this way; the Book must in early times have been attributed to Euclid, and the inference based upon this assumption was left uncorrected afterwards when it was recognised that Hypsicles was the author. The preface is worth quoting:

"Basilides of Tyre, O Protarchus, when he came to Alexandria and met my father, spent the greater part of his sojourn with him on account of their common interest in mathematics. And once, when

[1] The authorities for these statements quoted by Casiri and Ḥājī Khalfa are al-Kindī's tract *de instituto libri Euclidis* (al-Kindī died about 873) and a commentary by Qāḍīzāde ar-Rūmī (d. about 1440) on a book called *Ashkāl at-ta' sīs* (fundamental propositions) by Ashraf Shamsaddīn as-Samarqandī (*c.* 1276) consisting of elucidations of 35 propositions selected from the first books of Euclid. Naṣīraddīn likewise says that Euclid cut out two of 15 books of elements then existing and published the rest under his own name. According to Qāḍīzāde the king heard that there was a celebrated geometer named Euclid at *Tyre*: Naṣīraddīn says that he sent for Euclid of Ṭūs.

[2] So says the *Fihrist*. Suter (*op. cit.* p. 49) thinks that the author of the *Fihrist* did not suppose Apollonius *of Perga* to be the writer of the *Elements*, as later Arabian authorities did, but that he distinguished another Apollonius whom he calls "a carpenter." Suter's argument is based on the fact that the *Fihrist's* article on Apollonius (of Perga) says nothing of the *Elements*; and that it gives the three great mathematicians, Euclid, Archimedes and Apollonius, in the correct chronological order.

examining the treatise written by Apollonius about the comparison between the dodecahedron and the icosahedron inscribed in the same sphere, (showing) what ratio they have to one another, they thought that Apollonius had not expounded this matter properly, and accordingly they emended the exposition, as I was able to learn from my father. And I myself, later, fell in with another book published by Apollonius, containing a demonstration relating to the subject, and I was greatly interested in the investigation of the problem. The book published by Apollonius is accessible to all—for it has a large circulation, having apparently been carefully written out later—but I decided to send you the comments which seem to me to be necessary, for you will through your proficiency in mathematics in general and in geometry in particular form an expert judgment on what I am about to say, and you will lend a kindly ear to my disquisition for the sake of your friendship to my father and your goodwill to me."

The idea that Apollonius preceded Euclid must evidently have been derived from the passage just quoted. It explains other things besides. Basilides must have been confused with *βασιλεύς*, and we have a probable explanation of the "Alexandrian king," and of the "learned men who visited" Alexandria. It is possible also that in the "Tyrian" of Hypsicles' preface we have the origin of the notion that Euclid was born in Tyre. These inferences argue, no doubt, very defective knowledge of Greek: but we could expect no better from those who took the *Organon* of Aristotle to be "instrumentum musicum pneumaticum," and who explained the name of Euclid, which they variously pronounced as *Uclides* or *Icludes*, to be compounded of *Ucli* a key, and *Dis* a measure, or, as some say, geometry, so that *Uclides* is equivalent to the *key of geometry*!

Lastly the alternative version, given in brackets above, which says that Euclid made the *Elements* out of commentaries which he wrote on two books of Apollonius on conics and prolegomena added to the doctrine of the five solids, seems to have arisen, through a like confusion, out of a later passage[1] in Hypsicles' Book XIV.: "And this is expounded by Aristaeus in the book entitled 'Comparison of the five figures,' and by Apollonius in the second edition of his comparison of the dodecahedron with the icosahedron." The "doctrine of the five solids" in the Arabic must be the "Comparison of the five figures" in the passage of Hypsicles, for nowhere else have we any information about a work bearing this title, nor can the Arabians have had. The reference to the *two books* of Apollonius on *conics* will then be the result of mixing up the fact that Apollonius wrote a book on conics with the *second edition* of the other work mentioned by Hypsicles. We do not find elsewhere in Arabian authors any mention of a commentary by Euclid on Apollonius and Aristaeus: so that the story in the passage quoted is really no more than a variation of the fable that the *Elements* were the work of Apollonius.

[1] Heiberg's Euclid, vol. v. p. 6.

CHAPTER II.

EUCLID'S OTHER WORKS.

IN giving a list of the Euclidean treatises other than the *Elements*, I shall be brief: for fuller accounts of them, or speculations with regard to them, reference should be made to the standard histories of mathematics[1].

I will take first the works which are mentioned by Greek authors.

1. The *Pseudaria*.

I mention this first because Proclus refers to it in the general remarks in praise of the *Elements* which he gives immediately after the mention of Euclid in his summary. He says[2]: "But, inasmuch as many things, while appearing to rest on truth and to follow from scientific principles, really tend to lead one astray from the principles and deceive the more superficial minds, he has handed down methods for the discriminative understanding of these things as well, by the use of which methods we shall be able to give beginners in this study practice in the discovery of paralogisms, and to avoid being misled. This treatise, by which he puts this machinery in our hands, he entitled (the book) of Pseudaria, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of error with practical illustration. This book then is by way of cathartic and exercise, while the Elements contain the irrefragable and complete guide to the actual scientific investigation of the subjects of geometry."

The book is considered to be irreparably lost. We may conclude however from the connexion of it with the *Elements* and the reference to its usefulness for beginners that it did not go outside the domain of elementary geometry[3].

[1] See, for example, Loria, *Le scienze esatte nell' antica Grecia*, 1914, pp. 245—268; T. L. Heath, *History of Greek Mathematics*, 1921, I. pp. 421—446. Cf. Heiberg, *Litterargeschichtliche Studien über Euklid*, pp. 36—153; *Euclidis opera omnia*, ed. Heiberg and Menge, Vols. VI.—VIII.

[2] Proclus, p. 70, 1—18.

[3] Heiberg points out that Alexander Aphrodisiensis appears to allude to the work in his commentary on Aristotle's *Sophistici Elenchi* (fol. 25 *b*): "Not only those (ἔλεγχοι) which do not start from the principles of the science under which the problem is classed...but also those which do start from the proper principles of the science but in some respect admit a paralogism, e.g. the *Pseudographemata* of Euclid." Tannery (*Bull. des sciences math. et astr.* 2e Série, VI., 1882, 1ère Partie, p. 147) conjectures that it may be from this treatise that the same commentator got his information about the quadratures of the circle by Antiphon and

2. The *Data*.

The *Data* (δεδομένα) are included by Pappus in the *Treasury of Analysis* (τόπος ἀναλυόμενος), and he describes their contents[1]. They are still concerned with elementary geometry, though forming part of the introduction to higher analysis. Their form is that of propositions proving that, if certain things in a figure are given (in magnitude, in species, etc.), something else is given. The subject-matter is much the same as that of the planimetrical books of the *Elements*, to which the *Data* are often supplementary. We shall see this later when we come to compare the propositions in the *Elements* which give us the means of solving the general quadratic equation with the corresponding propositions of the *Data* which give the solution. The *Data* may in fact be regarded as elementary exercises in analysis.

It is not necessary to go more closely into the contents, as we have the full Greek text and the commentary by Marinus newly edited by Menge and therefore easily accessible[2].

3. The book *On divisions* (*of figures*).

This work (περὶ διαιρέσεων βιβλίον) is mentioned by Proclus[3]. In one place he is speaking of the conception or definition (λόγος) of *figure*, and of the divisibility of a figure into others differing from it in kind; and he adds: "For the circle is divisible into parts unlike in definition or notion (ἀνόμοια τῷ λόγῳ), and so is each of the rectilineal figures; this is in fact the business of the writer of the Elements in his Divisions, where he divides given figures, in one case into like figures, and in another into unlike[4]." "Like" and "unlike" here mean, not "similar" and "dissimilar" in the technical sense, but "like" or "unlike *in definition* or *notion*" (λόγῳ): thus to divide a triangle into triangles would be to divide it into "like" figures, to divide a triangle into a triangle and a quadrilateral would be to divide it into "unlike" figures.

The treatise is lost in Greek but has been discovered in the Arabic. First John Dee discovered a treatise *De divisionibus* by one Muḥammad Bagdadinus[5] and handed over a copy of it (in Latin) in 1563 to Commandinus, who published it, in Dee's name and his own, in 1570[6]. Dee did not himself translate the tract from the Arabic; he

Bryson, to say nothing of the lunules of Hippocrates. I think however that there is an objection to this theory so far as regards Bryson; for Alexander distinctly says that Bryson's quadrature did *not* start from the proper principles of geometry, but from some principles more general.

[1] Pappus, VII. p. 638.

[2] Vol. VI. in the Teubner edition of *Euclidis opera omnia* by Heiberg and Menge. A translation of the *Data* is also included in Simson's Euclid (though naturally his text left much to be desired).

[3] Proclus, p. 69, 4.

[4] *ibid.* 144, 22—26.

[5] Steinschneider places him in the 10th c. H. Suter (*Bibliotheca Mathematica*, IV$_3$, 1903, pp. 24, 27) identifies him with Abū (Bekr) Muḥ. b. 'Abdalbāqī al-Baġdādī, Qāḍī (Judge) of Māristān (*circa* 1070–1141), to whom he also attributes the *Liber judei* (? judicis) *super decimum Euclidis* translated by Gherard of Cremona.

[6] *De superficierum divisionibus liber Machometo Bagdadino adscriptus, nunc primum Ioannis Dee Londinensis et Federici Commandini Urbinatis opera in lucem editus*, Pisauri, 1570, afterwards included in Gregory's Euclid (Oxford, 1703).

found it in Latin in a MS. which was then in his own possession but was about 20 years afterwards stolen or destroyed in an attack by a mob on his house at Mortlake[1]. Dee, in his preface addressed to Commandinus, says nothing of his having *translated* the book, but only remarks that the very illegible MS. had caused him much trouble and (in a later passage) speaks of "the actual, very ancient, copy from which I *wrote out*..." (in ipso unde descripsi vetustissimo exemplari). The Latin translation of this tract from the Arabic was probably made by Gherard of Cremona (1114–1187), among the list of whose numerous translations a "liber divisionum" occurs. The Arabic original cannot have been a direct translation from Euclid, and probably was not even a direct adaptation of it; it contains mistakes and unmathematical expressions, and moreover does not contain the propositions about the division of a circle alluded to by Proclus. Hence it can scarcely have contained more than a fragment of Euclid's work.

But Woepcke found in a MS. at Paris a treatise in Arabic on the division of figures, which he translated and published in 1851[2]. It is expressly attributed to Euclid in the MS. and corresponds to the description of it by Proclus. Generally speaking, the divisions are divisions into figures of the same kind as the original figures, e.g. of triangles into triangles; but there are also divisions into "unlike" figures, e.g. that of a triangle by a straight line parallel to the base. The missing propositions about the division of a circle are also here: "to divide into two equal parts a given figure bounded by an arc of a circle and two straight lines including a given angle" and "to draw in a given circle two parallel straight lines cutting off a certain part of the circle." Unfortunately the proofs are given of only four propositions (including the two last mentioned) out of 36, because the Arabic translator found them too easy and omitted them. To illustrate the character of the problems dealt with I need only take one more example: "To cut off a certain fraction from a (parallel-) trapezium by a straight line which passes through a given point lying inside or outside the trapezium but so that a straight line can be drawn through it cutting both the parallel sides of the trapezium." The genuineness of the treatise edited by Woepcke is attested by the facts that the four proofs which remain are elegant and depend on propositions in the *Elements*, and that there is a lemma with a true Greek ring: "to apply to a straight line a rectangle equal to the rectangle contained by *AB*, *AC and deficient by a square*." Moreover the treatise is no fragment, but finishes with the words "end of the treatise," and is a well-ordered and compact whole. Hence we may safely conclude that Woepcke's is not only Euclid's own work but the whole of it. A restoration of the work, with proofs, was attempted by Ofterdinger[3], who however does not give Woepcke's props. 30, 31, 34, 35, 36. We have now a satisfactory restoration, with ample notes

[1] R. C. Archibald, *Euclid's Book on the Division of Figures with a restoration based on Woepcke's text and on the Practica geometriae of Leonardo Pisano*, Cambridge, 1915, pp. 4—9.

[2] *Journal Asiatique*, 1851, p. 233 sqq.

[3] L. F. Ofterdinger, *Beiträge zur Wiederherstellung der Schrift des Euklides über die Theilung der Figuren*, Ulm, 1853.

and an introduction, by R. C. Archibald, who used for the purpose Woepcke's text and a section of Leonardo of Pisa's *Practica geometriae* (1220)[1].

4. The *Porisms*.

It is not possible to give in this place any account of the controversies about the contents and significance of the three lost books of Porisms, or of the important attempts by Robert Simson and Chasles to restore the work. These may be said to form a whole literature, references to which will be found most abundantly given by Heiberg and Loria, the former of whom has treated the subject from the philological point of view, most exhaustively, while the latter, founding himself generally on Heiberg, has added useful details, from the mathematical side, relating to the attempted restorations, etc.[2] It must suffice here to give an extract from the only original source of information about the nature and contents of the *Porisms*, namely Pappus[3]. In his general preface about the books composing the *Treasury of Analysis* (τόπος ἀναλυόμενος) he says:

"After the Tangencies (of Apollonius) come, in three books, the Porisms of Euclid, [in the view of many] a collection most ingeniously devised for the analysis of the more weighty problems, [and] although nature presents an unlimited number of such porisms[4], [they have added nothing to what was written originally by Euclid, except that some before my time have shown their want of taste by adding to a few (of the propositions) second proofs, each (proposition) admitting of a definite number of demonstrations, as we have shown, and Euclid having given one for each, namely that which is the most lucid. These porisms embody a theory subtle, natural, necessary, and of considerable generality, which is fascinating to those who can see and produce results].

"Now all the varieties of porisms belong, neither to theorems nor problems, but to a species occupying a sort of intermediate position [so that their enunciations can be formed like those of either theorems or problems], the result being that, of the great number of geometers, some regarded them as of the class of theorems, and others of problems, looking only to the form of the proposition. But that the ancients knew better the difference between these three things is clear from the definitions. For they said that a theorem is that which is proposed with a view to the demonstration of the very thing proposed, a problem that which is thrown out with a view to the construction of the very thing proposed, and a porism that which is proposed with a view to the producing of the very thing proposed. [But this definition of the porism was changed by the more recent writers who could not produce everything, but used these elements

[1] There is a remarkable similarity between the propositions of Woepcke's text and those of Leonardo, suggesting that Leonardo may have had before him a translation (perhaps by Gherard of Cremona) of the Arabic tract.

[2] Heiberg, *Euklid-Studien*, pp. 56—79, and Loria, *op. cit.*, pp. 253—265.

[3] Pappus, ed. Hultsch, VII. pp. 648—660. I put in square brackets the words bracketed by Hultsch.

[4] I adopt Heiberg's reading of a comma here instead of a full stop.

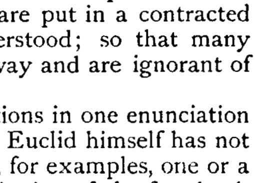
and proved only the fact that that which is sought really exists, but did not produce it[1] and were accordingly confuted by the definition and the whole doctrine. They based their definition on an incidental characteristic, thus: A porism is that which falls short of a locus-theorem in respect of its hypothesis[2]. Of this kind of porisms loci are a species, and they abound in the Treasury of Analysis; but this species has been collected, named and handed down separately from the porisms, because it is more widely diffused than the other species]. But it has further become characteristic of porisms that, owing to their complication, the enunciations are put in a contracted form, much being by usage left to be understood; so that many geometers understand them only in a partial way and are ignorant of the more essential features of their contents.

"[Now to comprehend a number of propositions in one enunciation is by no means easy in these porisms, because Euclid himself has not in fact given many of each species, but chosen, for examples, one or a few out of a great multitude[3]. But at the beginning of the first book he has given some propositions, to the number of ten, of one species, namely that more fruitful species consisting of loci.] Consequently, finding that these admitted of being comprehended in one enunciation, we have set it out thus:

> If, in a system of four straight lines[4] which cut each other two and two, three points on one straight line be given while the rest except one lie on different straight lines given in position, the remaining point also will lie on a straight line given in position[5].

[1] Heiberg points out that Props. 5—9 of Archimedes' treatise *On Spirals* are porisms in this sense. To take Prop. 5 as an example, DBF is a tangent to a circle with centre K. It is then possible, says Archimedes, to draw a straight line KHF, meeting the circumference in H and the tangent in F, such that

$$FH : HK < (\text{arc } BH) : c,$$



where c is the circumference of *any* circle. To prove this he assumes the following construction. E being any straight line greater than c, he says: let KG be parallel to DF, "and let the line GH equal to E be placed *verging* to the point B." Archimedes must of course have known how to effect this construction, which requires conics. But that it is *possible* requires very little argument, for if we draw any straight line BHG meeting the circle in H and KG in G, it is obvious that as G moves away from C, HG becomes greater and greater and may be made as great as we please. The "later writers" would no doubt have contented themselves with this consideration without actually *constructing* HG.

[2] As Heiberg says, this translation is made certain by a preceding passage of Pappus (p. 648, 1—3) where he compares two enunciations, the latter of which "falls short of the former in *hypothesis* but goes beyond it in *requirement*." E.g. the first enunciation requiring us, given three circles, to draw a circle touching all three, the second may require us, given only *two* circles (one less datum), to draw a circle touching them and *of a given size* (an extra requirement).

[3] I translate Heiberg's reading with a full stop here followed by πρὸς ἀρχῇ δὲ ὅμως [πρὸς ἀρχὴν (δεδομένον) Hultsch] τοῦ πρώτου βιβλίου....

[4] The four straight lines are described in the text as (the sides) ὑπτίου ἢ παρυπτίου, i.e. sides of two sorts of quadrilaterals which Simson tries to explain (see p. 120 of the *Index Graecitatis* of Hultsch's edition of Pappus).

[5] In other words (Chasles, p. 23; Loria, p. 256), if a triangle be so deformed that each of its sides turns about one of three points in a straight line, and two of its vertices lie on two straight lines given in position, the third vertex will also lie on a straight line.

"This has only been enunciated of four straight lines, of which not more than two pass through the same point, but it is not known (to most people) that it is true of any assigned number of straight lines if enunciated thus:

> If any number of straight lines cut one another, not more than two (passing) through the same point, and all the points (of intersection situated) on one of them be given, and if each of those which are on another (of them) lie on a straight line given in position—

or still more generally thus:

> if any number of straight lines cut one another, not more than two (passing) through the same point, and all the points (of intersection situated) on one of them be given, while of the other points of intersection in multitude equal to a triangular number a number corresponding to the side of this triangular number lie respectively on straight lines given in position, provided that of these latter points no three are at the angular points of a triangle (*sc.* having for sides three of the given straight lines)—each of the remaining points will lie on a straight line given in position[1].

"It is probable that the writer of the Elements was not unaware of this but that he only set out the principle; and he seems, in the case of all the porisms, to have laid down the principles and the seed only [of many important things], the kinds of which should be distinguished according to the differences, not of their hypotheses, but of the results and the things sought. [All the hypotheses are different from one another because they are entirely special, but each of the results and things sought, being one and the same, follow from many different hypotheses.]

"We must then in the first book distinguish the following kinds of things sought:

"At the beginning of the book[2] is this proposition:

> I. '*If from two given points straight lines be drawn meeting on a straight line given in position, and one cut off from a straight line given in position* (*a segment measured*) *to a given point on it, the other will also cut off from another* (*straight line a segment*) *having to the first a given ratio.*'

"Following on this (we have to prove)

> II. that such and such a point lies on a straight line given in position;
>
> III. that the ratio of such and such a pair of straight lines is given;"
>
> etc. etc. (up to XXIX.).

"The three books of the porisms contain 38 lemmas; of the theorems themselves there are 171."

[1] Loria (p. 256, *n.* 3) gives the meaning of this as follows, pointing out that Simson was the discoverer of it: "If a complete n-lateral be deformed so that its sides respectively turn about n points on a straight line, and $(n-1)$ of its $n(n-1)/2$ vertices move on as many straight lines, the other $(n-1)(n-2)/2$ of its vertices likewise move on as many straight lines: but it is necessary that it should be impossible to form with the $(n-1)$ vertices any triangle having for sides the sides of the polygon."

[2] Reading, with Heiberg, τοῦ βιβλίου [τοῦ ζ' Hultsch].

Pappus further gives lemmas to the *Porisms* (pp. 866—918, ed. Hultsch).

With Pappus' account of Porisms must be compared the passages of Proclus on the same subject. Proclus distinguishes two senses in which the word πόρισμα is used. The first is that of *corollary* where something appears as an incidental result of a proposition, obtained without trouble or special seeking, a sort of bonus which the investigation has presented us with[1]. The other sense is that of Euclid's *Porisms*[2]. In this sense[3] "*porism* is the name given to things which are sought, but need some finding and are neither pure bringing into existence nor simple theoretic argument. For (to prove) that the angles at the base of isosceles triangles are equal is a matter of theoretic argument, and it is with reference to things existing that such knowledge is (obtained). But to bisect an angle, to construct a triangle, to cut off, or to place—all these things demand the making of something; and to find the centre of a given circle, or to find the greatest common measure of two given commensurable magnitudes, or the like, is in some sort between theorems and problems. For in these cases there is no bringing into existence of the things sought, but finding of them, nor is the procedure purely theoretic. For it is necessary to bring that which is sought into view and exhibit it to the eye. Such are the porisms which Euclid wrote, and arranged in three books of Porisms."

Proclus' definition thus agrees well enough with the first, "older," definition of Pappus. A porism occupies a place between a theorem and a problem: it deals with something already *existing*, as a theorem does, but has to *find* it (e.g. the centre of a circle), and, as a certain operation is therefore necessary, it partakes to that extent of the nature of a problem, which requires us to construct or produce something not previously existing. Thus, besides III. 1 of the *Elements* and X. 3, 4 mentioned by Proclus, the following propositions are real porisms: III. 25, VI. 11—13, VII. 33, 34, 36, 39, VIII. 2, 4, X. 10, XIII. 18. Similarly in Archimedes *On the Sphere and Cylinder* I. 2—6 might be called porisms.

The enunciation given by Pappus as comprehending ten of Euclid's propositions may not reproduce the *form* of Euclid's enunciations; but, comparing the result to be proved, that certain points lie on straight lines given in position, with the *class* indicated by II. above, where the question is of such and such a point lying on a straight line given in position, and with other classes, e.g. (V.) that such and such a line is given in position, (VI.) that such and such a line verges to a given point, (XXVII.) that there exists a given point such that straight lines drawn from it to such and such (circles) will contain a triangle given in species, we may conclude that a usual form of a porism was "to prove that it is possible to find a point with such and such a property"

[1] Proclus, pp. 212, 14; 301, 22.

[2] *ibid.* p. 212, 12. "The term porism is used of certain problems, like the *Porisms* written by Euclid."

[3] *ibid.* pp. 301, 25 sqq.

or "a straight line on which lie all the points satisfying given conditions" etc.

Simson defined a porism thus: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem datis, sed quae ad ea quae data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam[1]."

From the above it is easy to understand Pappus' statement that *loci* constitute a large class of porisms. A *locus* is well defined by Simson thus: "Locus est propositio in qua propositum est datam esse demonstrare, vel invenire lineam aut superficiem cuius quodlibet punctum, vel superficiem in qua quaelibet linea data lege descripta, communem quandam habet proprietatem in propositione descriptam." Heiberg cites an excellent instance of a *locus* which is a *porism*, namely the following proposition quoted by Eutocius[2] from the *Plane Loci* of Apollonius:

"Given two points in a plane, and a ratio between unequal straight lines, it is possible to draw, in the plane, a circle such that the straight lines drawn from the given points to meet on the circumference of the circle have (to one another) a ratio the same as the given ratio."

A difficult point, however, arises on the passage of Pappus, which says that a porism is "that which, in respect of its hypothesis, falls short of a locus-theorem" (τοπικοῦ θεωρήματος). Heiberg explains it by comparing the porism from Apollonius' *Plane Loci* just given with Pappus' enunciation of the same thing, to the effect that, if from two given points two straight lines be drawn meeting in a point, and these straight lines have to one another a given ratio, the point will lie on either a straight line or a circumference of a circle given in position. Heiberg observes that in this latter enunciation something is taken into the hypothesis which was not in the hypothesis of the enunciation of the porism, viz. "that the ratio of the straight lines is the same." I confess this does not seem to me satisfactory: for there is no real difference between the enunciations, and the supposed difference in hypothesis is very like playing with words. Chasles says: "*Ce qui constitue le porisme est ce qui manque à l'hypothèse d'un théorème local* (en d'autres termes, le porisme est inférieur, par l'hypothèse, au théorème local; c'est-à-dire que quand quelques parties d'une proposition locale n'ont pas dans l'énoncé la détermination qui leur est propre, cette proposition cesse d'être regardée comme un théorème et devient un porisme)." But the subject still seems to require further elucidation.

While there is so much that is obscure, it seems certain (1) that the *Porisms* were distinctly part of higher geometry and not of elementary

[1] This was thus expressed by Chasles: "Le porisme est une proposition dans laquelle on demande de démontrer qu'une chose ou plusieurs choses sont *données*, qui, ainsi que l'une quelconque d'une infinité d'autres choses non données, mais dont chacune est avec des choses données dans une même relation, ont une certaine propriété commune, décrite dans la proposition."

[2] Commentary on Apollonius' *Conics* (vol. II. p. 180, ed. Heiberg).

geometry, (2) that they contained propositions belonging to the modern theory of transversals and to projective geometry. It should be remembered too that it was in the course of his researches on this subject that Chasles was led to the idea of *anharmonic ratios.*

Lastly, allusion should be made to the theory of Zeuthen[1] on the subject of the porisms. He observes that the only porism of which Pappus gives the complete enunciation, "If from two given points straight lines be drawn meeting on a straight line given in position, and one cut off from a straight line given in position (a segment measured) towards a given point on it, the other will also cut off from another (straight line a segment) bearing to the first a given ratio," is also true if there be substituted for the first given straight line a conic regarded as the "locus with respect to four lines," and that this extended porism can be used for completing Apollonius' exposition of that locus. Zeuthen concludes that the *Porisms* were in part by-products of the theory of conics and in part auxiliary means for the study of conics, and that Euclid called them by the same name as that applied to corollaries because they were corollaries with respect to conics. But there appears to be no evidence to confirm this conjecture.

5. The *Surface-loci* (τόποι πρὸς ἐπιφανείᾳ).

The two books on this subject are mentioned by Pappus as part of the *Treasury of Analysis*[2]. As the other works in the list which were on plane subjects dealt only with straight lines, circles, and conic sections, it is *a priori* likely that among the loci in this treatise (loci which are surfaces) were included such loci as were cones, cylinders and spheres. Beyond this all is conjecture based on two lemmas given by Pappus in connexion with the treatise.

(1) The first of these lemmas[3] and the figure attached to it are not satisfactory as they stand, but a possible restoration is indicated by Tannery[4]. If the latter is right, it suggests that one of the loci contained all the points on the elliptical parallel sections of a cylinder and was therefore an oblique circular cylinder. Other assumptions with regard to the conditions to which the lines in the figure may be subject would suggest that other loci dealt with were cones regarded as containing all points on particular elliptical parallel sections of the cones[5].

(2) In the second lemma Pappus states and gives a complete proof of the focus-and-directrix property of a conic, viz. that *the locus of a point whose distance from a given point is in a given ratio to its distance from a fixed line is a conic section, which is an ellipse, a parabola or a hyperbola according as the given ratio is less than, equal to, or greater than unity*[6]. Two conjectures are possible as to the application of this theorem in Euclid's *Surface-loci.* (*a*) It may have been used to prove that the locus of a point whose distance from a given straight

[1] *Die Lehre von den Kegelschnitten im Altertum*, chapter VIII.
[2] Pappus, VII. p. 636. [3] *ibid.* VII. p. 1004.
[4] *Bulletin des sciences math. et astron.*, 2e Série, VI. 149.
[5] Further particulars will be found in *The Works of Archimedes*, pp. lxii—lxiv, and in Zeuthen, *Die Lehre von den Kegelschnitten*, p. 425 sqq.
[6] Pappus, VII. pp. 1006—1014, and Hultsch's Appendix, pp. 1270—3.

line is in a given ratio to its distance from a given plane is a certain cone. (*b*) It may have been used to prove that the locus of a point whose distance from a given point is in a given ratio to its distance from a given plane is the surface formed by the revolution of a conic about its major or conjugate axis[1]. Thus Chasles may have been correct in his conjecture that the *Surface-loci* dealt with surfaces of revolution of the second degree and sections of the same[2].

6. The *Conics*.

Pappus says of this lost work: "The four books of Euclid's Conics were completed by Apollonius, who added four more and gave us eight books of Conics[3]." It is probable that Euclid's work was lost even by Pappus' time, for he goes on to speak of "Aristaeus, who wrote the *still extant* five books of Solid Loci connected with the conics." Speaking of the relation of Euclid's work to that of Aristaeus on conics regarded as loci, Pappus says in a later passage (bracketed however by Hultsch) that Euclid, regarding Aristaeus as deserving credit for the discoveries he had already made in conics, did not (try to) anticipate him or construct anew the same system. We may no doubt conclude that the book by Aristaeus on solid loci preceded Euclid's on conics and was, at least in point of originality, more important. Though both treatises dealt with the same subject-matter, the object and the point of view were different; had they been the same, Euclid could scarcely have refrained, as Pappus says he did, from attempting to improve upon the earlier treatise. No doubt Euclid wrote on the general theory of conics as Apollonius did, but confined himself to those properties which were necessary for the analysis of the *Solid Loci* of Aristaeus. The *Conics* of Euclid were evidently superseded by the treatise of Apollonius.

As regards the contents of Euclid's *Conics*, the most important source of our information is Archimedes, who frequently refers to propositions in conics as well known and not needing proof, adding in three cases that they are proved in the "elements of conics" or in "the conics," which expressions must clearly refer to the works of Aristaeus and Euclid[4]

Euclid still used the old names for the conics (sections of a right-angled, acute-angled, or obtuse-angled cone), but he was aware that an ellipse could be obtained by cutting a cone in any manner by a plane not parallel to the base (assuming the section to lie wholly between the apex of the cone and its base) and also by cutting a cylinder. This is expressly stated in a passage from the *Phaenomena* of Euclid about to be mentioned[5].

7. The *Phaenomena*.

This is an astronomical work and is still extant. A much inter-

[1] For further details see *The Works of Archimedes*, pp. lxiv, lxv, and Zeuthen, *l. c.*

[2] *Aperçu historique*, pp. 273—4.

[3] Pappus, VII. p. 672.

[4] For details of these propositions see my *Apollonius of Perga*, pp. xxxv, xxxvi.

[5] *Phaenomena*, ed. Menge, p. 6: "If a cone or a cylinder be cut by a plane not parallel to the base, the section is a section of an acute-angled cone, which is like a shield (θυρεός)."

polated version appears in Gregory's Euclid. An earlier and better recension is however contained in the MS. Vindobonensis philos. Gr. 103, though the end of the treatise, from the middle of prop. 16 to the last (18), is missing. The book, now edited by Menge[1], consists of propositions in *spheric* geometry. Euclid based it on Autolycus' work περὶ κινουμένης σφαίρας, but also, evidently, on an earlier textbook of *Sphaerica* of exclusively mathematical content. It has been conjectured that the latter textbook may have been due to Eudoxus[2].

8. The *Optics*.

This book needs no description, as it has been edited by Heiberg recently[3], both in its genuine form and in the recension by Theon. The *Catoptrica* published by Heiberg in the same volume is not genuine, and Heiberg suspects that in its present form it may be Theon's. It is not even certain that Euclid wrote *Catoptrica* at all, as Proclus may easily have had Theon's work before him and inadvertently assigned it to Euclid[4].

9. Besides the above-mentioned works, Euclid is said to have written the *Elements of Music*[5] (αἱ κατὰ μουσικὴν στοιχειώσεις). Two treatises are attributed to Euclid in our MSS. of the *Musici*, the κατατομὴ κανόνος, *Sectio canonis* (the theory of the intervals), and the εἰσαγωγὴ ἁρμονική (introduction to harmony)[6]. The first, resting on the Pythagorean theory of music, is mathematical, and the style and diction as well as the form of the propositions mostly agree with what we find in the *Elements*. Jan thought it genuine, especially as almost the whole of the treatise (except the preface) is quoted *in extenso*, and Euclid is twice mentioned by name, in the commentary on Ptolemy's *Harmonica* published by Wallis and attributed by him to Porphyry. Tannery was of the opposite opinion[7]. The latest editor, Menge, suggests that it may be a redaction by a less competent hand from the genuine Euclidean *Elements of Music*. The second treatise is not Euclid's, but was written by Cleonides, a pupil of Aristoxenus[8].

Lastly, it is worth while to give the Arabians' list of Euclid's works. I take this from Suter's translation of the list of philosophers and mathematicians in the *Fihrist*, the oldest authority of the kind that we possess[9]. "To the writings of Euclid belong further [in addition to the *Elements*]: the book of Phaenomena; the book of

[1] *Euclidis opera omnia*, vol. VIII., 1916, pp. 2—156.

[2] Heiberg, *Euklid-Studien*, p. 46; Hultsch, *Autolycus*, p. xii; A. A. Björnbo, *Studien über Menelaos' Sphärik* (*Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XIV. 1902), p. 56 sqq.

[3] *Euclidis opera omnia*, vol. VII. (1895).

[4] Heiberg, Euclid's *Optics*, *etc*. p. l.

[5] Proclus, p. 69, 3.

[6] Both treatises edited by Jan in *Musici Scriptores Graeci*, 1895, pp. 113—166, 167—207, and by Menge in *Euclidis opera omnia*, vol. VIII., 1916, pp. 157—183, 185—223.

[7] *Comptes rendus de l'Acad. des inscriptions et belles-lettres*, Paris, 1904, pp. 439—445. Cf. *Bibliotheca Mathematica*, VI3, 1905-6, p. 225, note 1.

[8] Heiberg, *Euklid-Studien*, pp. 52—55; Jan, *Musici Scriptores Graeci*, pp. 169—174.

[9] H. Suter, *Das Mathematiker-Verzeichniss im Fihrist* in *Abhandlungen zur Geschichte der Mathematik*, VI., 1892, pp. 1—87 (see especially p. 17). Cf. Casiri, I. 339, 340, and Gartz, *De interpretibus et explanatoribus Euclidis Arabicis*, 1823, pp. 4, 5.

Given Magnitudes [*Data*]; the book of Tones, known under the name of Music, not genuine; the book of Division, emended by Thābit; the book of Utilisations or Applications [*Porisms*], not genuine; the book of the Canon; the book of the Heavy and Light; the book of Synthesis, not genuine; and the book of Analysis, not genuine."

It is to be observed that the Arabs already regarded the book of Tones (by which must be meant the εἰσαγωγὴ ἁρμονική) as spurious. The book of Division is evidently the book on *Divisions* (*of figures*). The next book is described by Casiri as "liber de utilitate suppositus." Suter gives reason for believing the *Porisms* to be meant[1], but does not apparently offer any explanation of why the work is supposed to be spurious. The book of the Canon is clearly the κατατομὴ κανόνος. The book on "the Heavy and Light" is apparently the tract *De levi et ponderoso*, included in the Basel Latin translation of 1537, and in Gregory's edition. The fragment, however, cannot safely be attributed to Euclid, for (1) we have nowhere any mention of his having written on mechanics, (2) it contains the notion of specific gravity in a form so clear that it could hardly be attributed to anyone earlier than Archimedes[2]. Suter thinks[3] that the works on Analysis and Synthesis (said to be spurious in the extract) may be further developments of the *Data* or *Porisms*, or may be the interpolated proofs of *Eucl.* XIII. 1—5, divided into *analysis* and *synthesis*, as to which see the notes on those propositions.

[1] Suter, *op. cit.* pp. 49, 50. Wenrich translated the word as "utilia." Suter says that the nearest meaning of the Arabic word as of "porism" is *use*, *gain* (Nutzen, Gewinn), while a further meaning is explanation, observation, addition: a gain arising out of what has preceded (cf. Proclus' definition of the porism in the sense of a corollary).

[2] Heiberg, *Euklid-Studien*, pp. 9, 10.

[3] Suter, *op. cit.* p. 50.

MATHEMATICS

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